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INDUCIBLE PERIODIC HOMEOMORPHISMS OF TREE-LIKE CONTINUA[1]

BY

JUAN A. TOLEDO[2]

ABSTRACT. In this paper we prove that every periodic homeomorphism on a tree-like continuum can be strongly induced on an inverse sequence composed of a certain kind of graph that we call "bellows". We introduce the concepts of "#-graph" of a periodic homeomorphism and of "perfect" homeomorphism. A theorem concerning the parallel inducing of two periodic homeomorphisms having orbit spaces with the same multiplicity structure is also proved. The results are related to conjugacy and to the pseudo-arc.

1. Introduction. In this paper we are concerned with the problem of inducing periodic homeomorphisms defined on tree-like continua as inverse limits of homeomorphisms of graphs.

Many interesting examples of such homeomorphisms have been found in the past few years [**Br**, **Lew**, **T**] and most of them have been described as inverse limits. Certainly, this kind of description is very useful in the study of their properties and it results that it is highly desirable to be able to obtain such descriptions in terms of "nice" inverse sequences composed of "nice" spaces.

The purpose of this paper is to establish some criteria to determine when a situation is "nice" and to produce some results along those lines. The search will be oriented towards the study of conjugacy classes of periodic homeomorphisms on tree-like continua and in particular on the pseudo-arc (see [**B**]).

§§2 and 3 are devoted to developing the machinery to be used in §§4–6. This machinery is an elaboration of the techniques used by Fugate and McLean [**FM**].

In §4 we introduce the concept of a "spread cover" of a continuum M with respect to a periodic homeomorphism h. Covers of this type are finite open covers of M, invariant under h and with the property that its elements intersect each other in a convenient way. Theorem 4.13 guarantees the existence of spread covers of arbitrarily small mesh. This allows us to construct, in the case when the orbit space is tree-like, one-dimensional invariant covers having "bellows" (see Definition 2.13) as nerves.

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Using a theorem by Kaul [**K**], we prove in Theorem 5.3 that every periodic homeomorphism whose orbit space is tree-like can be induced as an inverse limit of homeomorphisms of bellows. Later in §5, motivated by a question by B. Brechner (Question 5.19), we discuss other forms of inducibility and as a result, the notion of "perfect" homeomorphism is introduced.

§6 is more oriented to the study of conjugacy through a result concerning parallel inducibility of two periodic homeomorphisms on tree-like continua with the same orbit space (Theorem 6.3).

The relationship between the pseudo-arc and the concepts introduced is closely analyzed in §§5 and 6, where several questions of interest are posed. Producing answers to these questions would represent an enormous contribution to the study of periodic homeomorphisms on the pseudo-arc.

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2. Preliminaries. All spaces are topological metric spaces. By a *continuum* we mean a connected, compact metric space.

DEFINITION 2.1. An *open cover* $\mathcal{U}$ of a metric space X is a collection $\mathcal{U} = \{U_i\}$ of open sets of X such that $X \subset \bigcup \{U_i\}$. All covers are always understood as *minimal*, i.e., each element of the cover contains a point of X not contained in any other element of the cover. If V is a subset of X and $\mathcal{U}$ is a cover, $V < \mathcal{U}$ means that V is contained in some element of $\mathcal{U}$ and is read "V refines $\mathcal{U}$". Similarly, if $\mathcal{V}$ is a cover, $\mathcal{V} < \mathcal{U}$, "$\mathcal{V}$ refines $\mathcal{U}$", means that $V < \mathcal{U}$ for every $V \in \mathcal{V}$. We will denote by $\mathfrak{N}(U)$ the *nerve* of U.

DEFINITION 2.2. A continuum is called *tree-like* (*chainable*) iff it has covers of arbitrarily small mesh whose nerves are trees (arcs).

DEFINITION 2.3. If Γ is a graph, vrt Γ and edg Γ will denote the sets of vertices and edges of Γ, respectively. The edge between v_1 and $v_2 \in \operatorname{vrt} \Gamma$ is denoted by $(v_1, v_2) \in \operatorname{edg} \Gamma$. Then (v_1, v_2) refers to the set $\{v_1, v_2\}$ rather than to the ordered pair (see Remark 2.4). A graph without cycles is a *tree* whereas a tree without points of order three or above is a *chain*. All graphs are finite. As topological spaces, graphs are finite one-dimensional simplical complexes.

REMARK 2.4. When defining a graph Γ we will often specify edg Γ by saying $(x_1, x_2) \in \operatorname{edg} \Gamma$ iff $P(x_1, x_2)$, where $P(x_1, x_2)$ is a certain property of the ordered pair (x_1, x_2). What we really mean is $(x_1, x_2) \in \operatorname{edg} \Gamma$ iff $P(x_1, x_2)$ or $P(x_2, x_1)$.

REMARK 2.5. If K is a finite partially ordered set under α, it can be made into a graph by defining vrt $K = K$ and $(x_1, x_2) \in \operatorname{edg} K$ iff $x_1 \neq x_2$, $x_1 \,\alpha\, x_2$ and $x_1 \,\alpha\, x \,\alpha\, x_2$ implies $x = x_1$ or $x = x_2$.

DEFINITION 2.6. The set of positive integers is denoted by N^+. A set $\{n, m\} \subset N^+$ is called *divisible* iff either $n \mid m$ or $m \mid n$.

DEFINITION 2.7. A #*-graph* ("number graph") is a pair (Γ, σ) where Γ is a graph and σ is a function $\sigma\colon \Gamma \to N^+$. If Γ is a tree or a chain, it is called a #*-tree* or a #*-chain* (see Figure 2.1). Very often, we will refer only to Γ whenever σ is understood.

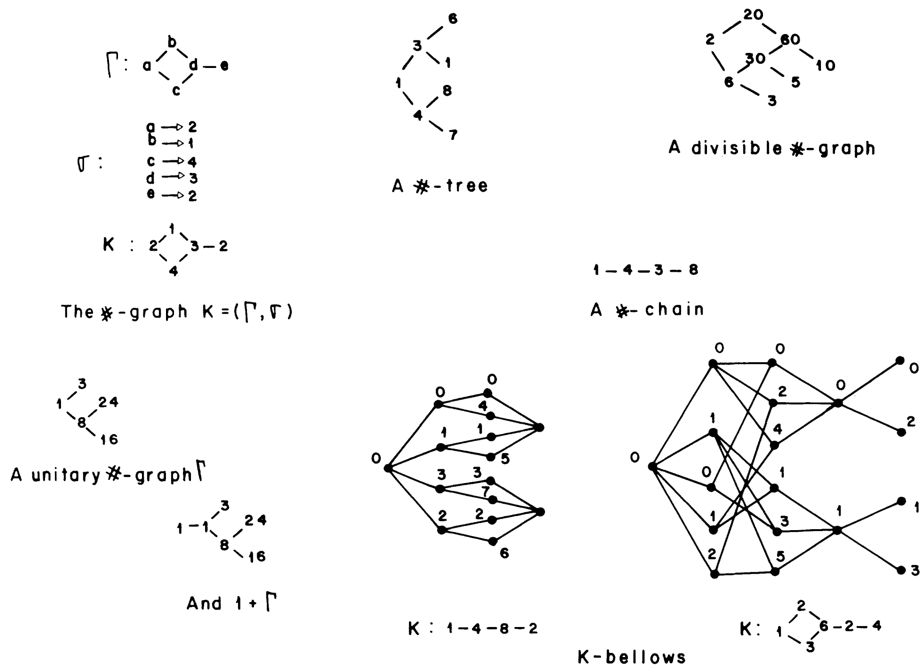


FIGURE 2.1

DEFINITION 2.8. A #-graph (Γ, σ) is *divisible* iff whenever $(a, b) \in \operatorname{edg} \Gamma$, the set $\{\sigma(a), \sigma(b)\}$ is divisible. In this case, Γ can be partially ordered by $x_1 \;\alpha\; x_2$ iff $(x_1, x_2) \in \operatorname{edg} \Gamma$ and $\sigma(x_1) \mid \sigma(x_2)$ or $x_1 < y_1 < y_2 < \cdots < y_k < x_2$ where (x_1, y_1), (y_i, y_{i+1}), (y_k, x_2) are edges of Γ and $\sigma(x_1) \mid \sigma(y_1) \mid \sigma(y_2) \mid \cdots \mid \sigma(y_k) \mid \sigma(x_2)$. A divisible #-graph is *unitary* if $\sigma(x_0) = 1$ where x_0 is unique minimal under the partial ordering. In this case $1 + \Gamma$ denotes the #-graph such that $\operatorname{vrt}(1 + \Gamma) = \Gamma \cup \{*\}$, $\sigma(*) = 1$ and $(*, x) \in \operatorname{edg}(1 + \Gamma)$ iff $x = x_0$ (see Figure 2.1).

DEFINITION 2.9. Let M be a continuum. A homeomorphism $h\colon M \to M$ is *periodic* iff there exists an integer $n > 0$ such that $h^n(x) = x$ for all $x \in M$. The *period of* h is the minimum integer n with the property. Assume now that h is periodic. If $x \in M$, the *order of* x, $\operatorname{ord} x$, is the minimum positive integer m such that $h^m(x) = x$. The *orbit of* x is $O(x) = \{h(x), h^2(x), \ldots, h^{\operatorname{ord} x}(x)\}$. The set of orbits of points of M forms a partition of M and when given the quotient topology becomes the *orbit space of M under h*, denoted M_h. Denote by $\pi\colon M \to M_h$ the projection map. If $x \in M_h$, the *multiplicity of* x, $\operatorname{mult} x$, is the number of elements of the orbit represented by x; that is, the cardinality of $\pi^{-1}(x)$.

LEMMA 2.10. *If h is periodic then π is open and closed, and if $\{U_i\}$ is a finite family of subsets of M_h then $\bigcap\{U_i\} \neq \varnothing$ iff $\bigcap\{\pi^{-1}(U_i)\} \neq \varnothing$.*

PROOF. Let h have period n and let U be an open (resp. closed) set of M. Then $\pi^{-1}\pi(U) = \bigcup\{h^i(U) \mid i = 0, \ldots, n-1\}$ which is open (resp. closed) since $h^i(U)$ is open (resp. closed). Then $\pi(U)$ is open (resp. closed). Sufficiency of the second statement follows from $\pi(\bigcap\{\pi^{-1}(U_i)\}) \subset \bigcap\{U_i\}$. For necessity, let $x \in \bigcap\{U_i\}$. Then there exists, for every i, an $x_i \in \pi^{-1}(U_i)$ such that $\pi(x_i) = x$. But then $O(x_i) \subset \pi^{-1}(U_i)$ for all i and since $O(x_i) = O(x_j)$, $\bigcap\{\pi^{-1}(U_i)\} \neq \varnothing$. $\square$

DEFINITION 2.11. Let $h\colon M \to M$ be a periodic homeomorphism on a continuum M. Let $\Gamma_h = \{i \in N^+ \mid i = \operatorname{ord} x \text{ for some } x \in M\}$. For each $i \in \Gamma_h$ let $F_i = \{x \in M \mid \operatorname{ord} x \mid i\}$. If M is tree-like, each F_i is a nonempty continuum **[FM]**. Furthermore, $F_i \subset F_j$ iff $i \mid j$ and $\bigcup\{F_i \mid i \in \Gamma_h\} = M$. Hence Γ_h and $\{F_i\}$, partially ordered by divisibility and containment, respectively, can be thought of as isomorphic graphs (see Remark 2.5). The *$\#$-graph of h*, denoted also by Γ_h, is the pair (Γ_h, σ) where $\sigma(i) = i$. A $\#$-graph Γ is *admissible* iff $\Gamma = \Gamma_h$ for some h periodic on some tree-like continuum M. The homeomorphism h is *divisible* iff its $\#$-graph is a chain. In this case $\Gamma_h = \{n_1 < n_2 < \cdots < n_p\}$ and $F_{n_1} \subset F_{n_2} \subset \cdots \subset F_{n_p} = M$. If M is tree-like then Γ_h is unitary.

LEMMA 2.12. *Let h be a periodic homeomorphism on a tree-like continuum M. Then Γ_h is divisible (as a $\#$-graph) and connected (as a topological space).*

PROOF. That Γ_h is divisible derives directly from its definition.

Suppose Γ_h is not connected. Then $\Gamma_h = A_1 \cup A_2$ where A_1 and A_2 are subgraphs of Γ_h such that no edge of Γ_h connects them. Let $M_i = \bigcup\{F_j \mid j \text{ is a maximal element of } A_i \text{ under the partial ordering}\}$ (see Definition 2.8). Then M_1 and M_2 are closed and $M = M_1 \cup M_2$. If $x \in M_1 \cap M_2$ with $\operatorname{ord} x = k$, then there are $j_1 \in A_1$ and $j_2 \in A_2$ minimal with the property that $k \mid j_1$ and $k \mid j_2$. Thus (k, j_1) and (k, j_2) are in $\operatorname{edg} \Gamma_h$, so A_1 and A_2 are connected by an edge, a contradiction. Thus, $M_1 \cap M_2 = \varnothing$ so M is not connected, again a contradiction. $\square$

DEFINITION 2.13. Let (K, σ) be a divisible $\#$-graph. A graph Γ is said to be a *(K, σ)-bellows* or simply a *K-bellows* iff

(i) $\operatorname{vrt} \Gamma = \bigcup\{V_k \mid k \in \operatorname{vrt} K\}$ where $V_k = \{v_{k1}, \ldots, v_{k\sigma(k)}\}$ and

(ii) $(v_{ki}, v_{lj}) \in \operatorname{edg} \Gamma$ iff

(a) $(k, l) \in \operatorname{edg} K$,

(b) $i \equiv j \bmod(\min\{\sigma(l), \sigma(k)\})$ (see Figure 2.1).

REMARK 2.14. It is not hard to prove that for a fixed divisible $\#$-graph K, any two K-bellows are isomorphic. Thus we can talk of *the* K-bellows.

DEFINITION 2.15. Let (K, σ) be a divisible $\#$-chain and let $k, l, m \in \operatorname{vrt} K$ be such that $(k, l), (l, m) \in \operatorname{edg} K$. We say that the triple k-l-m is a *redundancy* iff either $\sigma(k) < \sigma(l) = \sigma(m)$ or $\sigma(k) = \sigma(l) > \sigma(m)$. The divisible $\#$-chain (K', σ) obtained from (K, σ) by *eliminating the redundancy k-l-m* is the $\#$-chain such that $\operatorname{vrt} K' = \operatorname{vrt} K - \{l\}$, $\operatorname{edg} K' = (\operatorname{edg} K - \{(k, l), (l, m)\}) \cup \{(k, m)\}$ and σ is the same function restricted. A divisible $\#$-chain is *reduced* iff it does not contain any redundancy.

LEMMA 2.16. *For any divisible $\#$-chain Γ there exists a unique reduced $\#$-chain that can be obtained from Γ by eliminating a finite number of redundancies.*

PROOF. Given any divisible $\#$-chain, it contains only a finite number of redundancies. In whatever order they are eliminated, the same $\#$-chain is obtained and this is reduced. $\square$

DEFINITION 2.17. The reduced $\#$-chain obtained in Lemma 2.16 is called the *reduced $\#$-chain of* Γ and it is denoted by $\tilde{\Gamma}$.

LEMMA 2.18. *If $\tilde{\Gamma}$ is the reduced $\#$-chain of Γ, then the Γ-bellows and the $\tilde{\Gamma}$-bellows are homeomorphic.*

PROOF. Let $\Gamma = (K, \sigma)$. It is enough to prove that the Γ-bellows Φ and the Γ'-bellows Φ' are homeomorphic, where Γ' is obtained from Γ by eliminating one redundancy k-l-m, where $\sigma(k) < \sigma(l) = \sigma(m)$.

There are exactly $\sigma(m)$ edges in Φ' going from V_k to V_m, namely, $\{(v_{ki}, v_{mj}) \mid i = 1, \ldots, \sigma(m);\ i \equiv j \bmod \sigma(k)\}$. Select a point $V_{l'i}$ in the interior of each of the edges (v_{ki}, v_{mj}). Define a function ψ: $\operatorname{vrt} \Phi \to \operatorname{vrt} \Phi' \cup \{v_{l'i}\}$ by $\psi(v_{xi}) = v_{xi}$ if $x \neq l$ and $\psi(v_{li}) = v_{l'i}$. Extend this one-to-one correspondence linearly to the whole set Φ to obtain the desired homeomorphism. $\square$

LEMMA 2.19. *Let (K, σ) be a divisible $\#$-chain such that σ is monotone under the total order of K. Then the (K, σ)-bellows is a tree.*

PROOF. Use induction on the number of vertices of K. If K has one vertex, the lemma is trivial.

Suppose that $K = \{v_1 < v_2 < \cdots < v_n\}$. Then $\sigma|_{K-\{v_n\}}$ is also monotone so, by the induction hypothesis, the $(K - \{v_n\}, \sigma|_{K-\{v_n\}})$-bellows is a tree. Since $\sigma(v_{n-1}) \mid \sigma(v_n)$, for each i such that $0 \leqslant i < \sigma(v_n)$ there exists exactly one integer j, $0 \leqslant j < \sigma(v_{n-1})$, such that $i \equiv j \bmod \sigma(v_{n-1})$. Therefore, out of every vertex generated by v_n there is exactly one edge. Then the (K, σ)-bellows is a tree. $\square$

LEMMA 2.20. *Let (Γ, σ) be a $\#$-chain. Then (Γ, σ) is admissible iff Γ is connected, unitary and divisible and σ is monotone.*

PROOF. By Lemma 2.12, one way is clear. So let us suppose that Γ is connected, unitary and divisible and σ is monotone.

Consider the (Γ, σ)-bellows K. Define a function h: $\operatorname{vrt} K \to \operatorname{vrt} K$ by $h(v_{xi}) = v_{x(i+1 \bmod \sigma(x))}$, $x \in \Gamma$. The function h is a permutation on $\operatorname{vrt} K$ that extends by linearity to a homeomorphism h: $K \to K$. Let n be the maximal element of $\sigma(\Gamma)$. Then h is periodic of period n.

By Lemma 2.19 and since Γ is unitary and connected, K is a tree-like continuum.

Now consider Γ_h. In order to prove that $\Gamma = \Gamma_h$, it is sufficient to prove that $\Gamma_h = \{i \mid i = \operatorname{ord} x \text{ for some } x \in K\}$ equals $\sigma(\Gamma)$. If $i \in \sigma(\Gamma)$, then $\sigma(x) = i$ for some $x \in \operatorname{vrt} \Gamma$. The point v_{x0} has order $\sigma(x) = i$ so $i \in \Gamma_h$. Conversely, if $i \in \Gamma_h$, then $\operatorname{ord} y = i$ for some $y \in K$. If $y \in \operatorname{vrt} K$ then $y = v_{xj}$ for some $x \in \Gamma$ and then $\sigma(x) = i$, so $i \in \sigma(\Gamma)$. If $y \notin \operatorname{vrt} K$, then the whole edge containing it has order i. Thus one vertex of it must also have order i. Thus $\Gamma_h = \sigma(\Gamma)$. $\square$

REMARK 2.21. The homeomorphism described in the proof above constitutes an example of a perfect homeomorphism as defined in Definition 5.7.

3. Lifts. Throughout this section h: $M \to M$ will denote a periodic homeomorphism of a continuum M onto itself, M_h its orbit space and π: $M \to M_h$ the projection map.

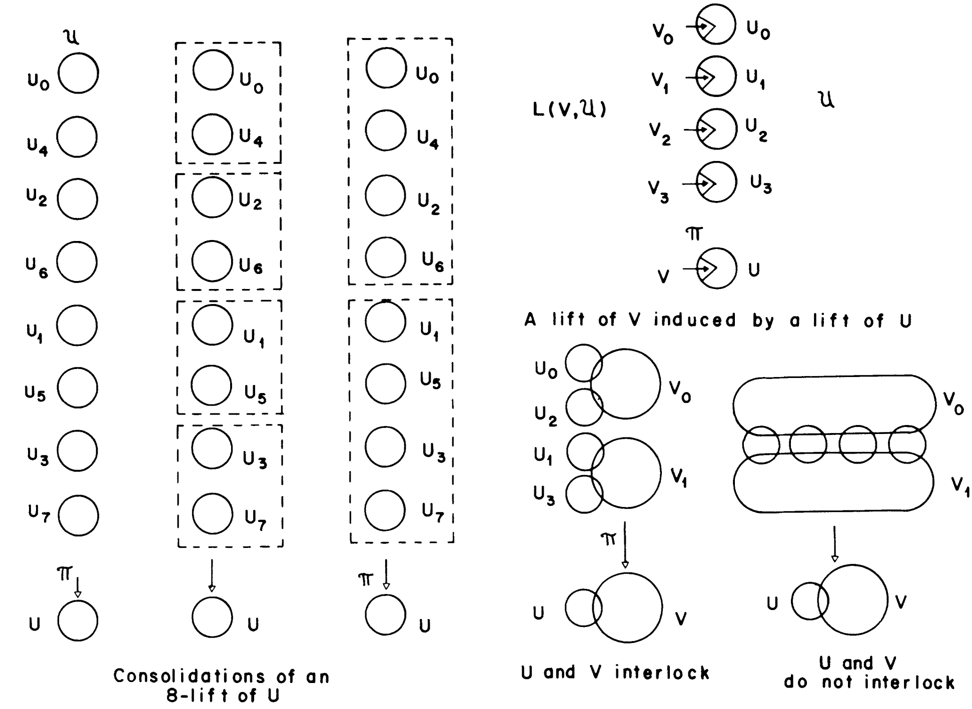


FIGURE 3.1

DEFINITION 3.1. Let U be an open set in M_h. A *lift of order n* or an *n-lift of U* is a collection of open sets of M, $\mathfrak{U} = \{U_i \mid i = 0,\dots,n-1\}$ such that

(i) $U_i \cap U_j = \varnothing$ iff $i \neq j$,

(ii) $h^s(U_i) = U_{i+s \bmod n}$,

(iii) $\pi^{-1}(U) = \cup\{U_i \mid i = 0,\dots,n-1\}$.

We write ord $\mathfrak{U} = n$ (see Figure 3.1).

REMARK 3.2. The labeling of the elements only specifies the way they are circularly ordered. It is therefore immaterial from what element the labeling starts. On many occasions we will change the starting point if it is more convenient.

There may be several different lifts of the same order for a fixed open set U as it is illustrated in the following

EXAMPLE 3.3. Let $U = Z \cup W$ where Z and W are open disjoint sets of M_h. Suppose that Z and W have 2-lifts $\{Z_1, Z_2\}$ and $\{W_1, W_2\}$. Then $\{Z_1 \cup W_1, Z_2 \cup W_2\}$ and $\{Z_1 \cup W_2, Z_2 \cup W_1\}$ are two different 2-lifts of U.

DEFINITION 3.4. Let $\mathfrak{U} = \{U_i \mid i = 0,\dots,n-1\}$ be an n-lift of $U \subset M_h$. Let $m \mid n$. The *m-consolidation of* $\mathfrak{U}$, denoted $\mathfrak{U}_m$, is defined by $\mathfrak{U}_m = \{V_i \mid i = 0,\dots,m-1\}$ where $V_i = \cup\{U_k \mid k \equiv i \bmod m\}$.

LEMMA 3.5. *If $\mathfrak{U}$ is an n-lift of U and $m \mid n$, then $\mathfrak{U}_m$ is an m-lift of U.*

PROOF. (i) If $V_i \cap V_j \neq \varnothing$ for $0 \leq i, j \leq m-1$, then, since the U_i's are pairwise disjoint, $U_k \subset V_i \cap V_j$ for some k such that $k \equiv i \bmod m$ and $k = j \bmod m$. But this implies $i = j$.

(ii) Since $m \mid n$, $p \equiv q \bmod m$ iff $p \equiv (q \bmod n) \bmod m$.
Thus we have

$$\begin{aligned} h^s(V_i) &= \bigcup \{U_{k+s \bmod n} \mid k \equiv i \bmod m\} = \bigcup \{U_l \mid l \equiv i + s \bmod m\} \\ &= \bigcup \{U_l \mid l \equiv (i + s \bmod n) \bmod m\} = V_{i+s \bmod n}. \end{aligned}$$

(iii) Clear. □

LEMMA 3.6. *If $\mathfrak{U}$ is an n-lift of U and $l \mid m \mid n$ then $\mathfrak{U}_m < \mathfrak{U}_l$.*

PROOF. Let $\mathfrak{U}_m = \{W_i \mid i = 0,\dots,m-1\}$ and $\mathfrak{U}_l = \{V_i \mid i = 0,\dots,l-1\}$. If $U_k \subset W_i$, then $k \equiv i \bmod m$ so $k \equiv i \bmod l$ and thus $k \equiv (i \bmod l) \bmod l$. From here we conclude that $U_k \subset V_{i_{\bmod l}}$ so $W_i \subset V_{i \bmod l}$ (see Figure 3.1).

LEMMA 3.7. *If $\mathfrak{U}$ is an m-lift of U and $l \mid m$ then $(\mathfrak{U}_m)_l = \mathfrak{U}_l$.*

PROOF. Let $\mathfrak{U}_m = \{V_i \mid i = 0,\dots,m-1\}$ where $V_i = \bigcup\{U_k \mid k \equiv i \bmod m\}$. By definition, $(\mathfrak{U}_m)_l = \{W_j \mid j = 1,\dots,l\}$ where

$$\begin{aligned} W_j &= \bigcup \{V_i \mid i \equiv j \bmod l\} = \bigcup \{U_k \mid k \equiv i \bmod m,\ i \equiv j \bmod l\} \\ &= \bigcup \{U_k \mid k \equiv j \bmod l\}. \end{aligned}$$

But this is nothing more than the jth element of $\mathfrak{U}_l$. □

There is an obvious connection between lifts and cyclic groups.

An n-lift can be identified with Z_n where s applications of h correspond to adding the number s modulo n. This way, the consolidations correspond to the quotient groups of $Z_n(\mathfrak{U}_m \leftrightarrow Z_n/Z_{n/m})$ and the last lemma derives from the fact that the set of quotients of Z_n constitutes a lattice isomorphic to the set of divisors of n.

For the sake of simplicity it is preferable not to invoke this connection and include direct proofs of the results even though they resemble very much the corresponding ones for groups.

LEMMA 3.8. *Let $\mathfrak{U} = \{U_i\}$ be an n-lift of U. Then $h^s(U_i) = U_i$ iff $n \mid s$.*

PROOF. $h^s(U_i) = U_{i+s \bmod n}$ *so $i + s \bmod n = i$.* This implies $s \equiv 0 \bmod n$. □

LEMMA 3.9. *Let $\mathfrak{U} = \{U_i\}$ be an n-lift of U. If $x \in \bigcup\{U_i\}$ then $n \mid \operatorname{ord} x$. If $x \in U$ then $n \mid \operatorname{mult} x$. The same is true for $x \in \bigcup\{\overline{U}_i\}$ in the case that $\overline{U}_i \cap \overline{U}_j = \varnothing$ iff $i \neq j$.*

PROOF. If $x \in \bigcup\{U_i\}$ then $x \in U_i$ for some i. Apply the previous lemma to $h^{\operatorname{ord} x}(U_i) = U_i$. If $x \in U$ then there exists $x' \in \bigcup\{U_i\}$ such that $\operatorname{ord} x' = \operatorname{mult} x$. Suppose that $\overline{U}_i \cap \overline{U}_j = \varnothing$ iff $i \neq j$. Then $h^s(\overline{U}_i) = \overline{U}_{i+s \bmod n}$. Proceed using similar arguments. □

DEFINITION 3.10. Let $\mathfrak{U} = \{U_i \mid i = 0,\dots,n-1\}$ be an n-lift of U. If $V \subset U$ is an open set, we define $L(V, \mathfrak{U})$, *the lift of V induced by $\mathfrak{U}$*, by $L(V, \mathfrak{U}) = \{U_i \cap \pi^{-1}(V) \mid i = 0,\dots,n-1\}$. If $m \mid n$ then $L_m(V, \mathfrak{U})$ denotes the m-consolidation of $L(V, \mathfrak{U})$. Since $h(U_i \cap \pi^{-1}(V)) = U_{i+1 \bmod n} \cap \pi^{-1}(V)$, this is well defined (see Figure 3.1).

LEMMA 3.11. *Let $\mathfrak{U}$ be an n-lift of U and $V \subset U$. For $m \mid n$, $L_m(V, \mathfrak{U}) = L(V, \mathfrak{U}_m)$.*

PROOF. Both lifts are of order m. The ith element of $L_m(V, \mathfrak{U})$ is

$$\bigcup \left\{ U_k \cap \pi^{-1}(V) \mid k \equiv i \bmod m \right\}.$$

But this is equal to $(\bigcup \{U_k \mid k \equiv i \bmod m\}) \cap \pi^{-1}(V)$ which is the ith element of $L(V, \mathfrak{U}_m)$. $\square$

LEMMA 3.12. *Let $V \subset U$ and let $\mathfrak{V}$ be an m-lift of V and $\mathfrak{U}$ an n-lift of U such that $\mathfrak{V} < \mathfrak{U}$. Then $L(V, \mathfrak{U}) = \mathfrak{V}_n$.*

PROOF. Assume that $V_0 \subset U_0$ (see Remark 3.2). Then $h^m(V_0) = V_0 \subset U_0$ so $h^m(U_0) = U_0$ and then, by Lemma 3.8, $n \mid m$. Let $0 \leqslant k \leqslant m-1$ and $0 \leqslant i \leqslant n-1$. Suppose $k \equiv i \bmod n$. Then $h^k(V_0) \subset h^k(U_0)$ implies $V_k \subset U_{k \bmod n}$, so $V_k \subset U_i$.

Conversely, if $V_k \subset U_i$ then $h^{-k}(V_k) \subset h^{-k}(U_i)$ so $V_0 \subset U_{i-k \bmod n}$ and, since $V_0 \subset U_0$, $i - k \bmod n = 0$ or $i \equiv k \bmod n$. Therefore $V_k \subset U_i$ iff $k \equiv i \bmod n$. Now, observe that the ith element of $L(V, \mathfrak{U})$ is $\pi^{-1}(V) \cap U_i$. But this is exactly $\{V_k \mid k \equiv i \bmod n\}$ which is the ith element of $\mathfrak{V}_n$. $\square$

The necessity of the condition $\mathfrak{V} < \mathfrak{U}$ can be easily illustrated as follows: Suppose that $\mathfrak{V} \not< \mathfrak{U}$ and that $n = m$. Then $\mathfrak{V}_n = \mathfrak{V}$ and since $L(V, \mathfrak{U}) < \mathfrak{U}$, $L(V, \mathfrak{U}) \neq \mathfrak{V}_n$.

LEMMA 3.13. *Let $W \subset V \subset U$ and $\mathfrak{U}$ a lift of U. Then $L(W, L(V, \mathfrak{U})) = L(W, \mathfrak{U})$.*

PROOF. Let $L(V, \mathfrak{U}) = \{T_i\}$. The lemma follows from the fact that $\pi^{-1}(W) \cap T_i = \pi^{-1}(W) \cap \pi^{-1}(V) \cap U_i = \pi^{-1}(W) \cap U_i$. $\square$

LEMMA 3.14. *Let $W \subset V \subset U$, $\mathfrak{V}$ an m-lift of V and $\mathfrak{U}$ an n-lift of U such that $\mathfrak{V} < \mathfrak{U}$. Then*

(i) *$L(W, \mathfrak{U}) = L_n(W, \mathfrak{V})$.*

(ii) *If $p \mid n \mid m$ then $L_p(W, \mathfrak{V}) = L_p(W, \mathfrak{U})$.*

(iii) *If $n \mid p \mid m$ then $L_p(W, \mathfrak{V}) < L(W, \mathfrak{U})$.*

PROOF Use the three previous lemmas to prove (i) by means of $L_n(W, \mathfrak{V}) = L(W, \mathfrak{V}_n) = L(W, L(V, \mathfrak{U})) = L(W, \mathfrak{U})$.

Use Lemma 3.7 and (i) to verify (ii) as follows:

$$L_p(W, \mathfrak{U}) = (L_n(W, \mathfrak{V}))_p = L_p(W, \mathfrak{V}).$$

Finally, (iii) follows from Lemma 3.6 and part (i). $\square$

Standing Notation 3.15. Up to the end of the proof of Corollary 3.23 we will assume that $U, V \subset M_h$, $\mathfrak{U}$ is an n-lift of U and $\mathfrak{V}$ is an m-lift of V. Also, any time $\mathfrak{U}$ *and* $\mathfrak{V}$ *are related*, we will assume (see Remark 3.2) that we have renumbered the elements of the lifts in such a way that U_0 *and* V_0 *are also related*. For example, if $\mathfrak{U} \cap \mathfrak{V} \neq \varnothing$ then $U_0 \cap V_0 \neq \varnothing$; if $\mathfrak{U} \subset \mathfrak{V}$ then $U_0 \subset V_0$, etc.

LEMMA 3.16. *Suppose that $U \cap V \neq \varnothing$. Then every U_i intersects at least one V_j and every V_i intersects at least one U_j.*

PROOF. Recall from Lemma 2.10 and Standing Notation 3.15 that $U \cap V \neq \varnothing$ iff $\mathfrak{U} \cap \mathfrak{V} \neq \varnothing$ and that $U_0 \cap V_0 \neq \varnothing$. Then $U_i = h^i(U_0)$ intersects $V_{i \bmod m}$ and $V_j = h^j(V_0)$ intersects $U_{j \bmod n}$. $\square$

DEFINITION 3.17. Suppose $U \cap V \neq \varnothing$. We say that $\mathfrak{U}$ *and* $\mathfrak{V}$ *interlock* iff

(i) the set $\{m, n\}$ is divisible,

(ii) $U_i \cap V_j \neq \varnothing$ iff $i \equiv j \bmod(\min\{n, m\})$ (see Figure 3.1).

The following lemma is a direct consequence of the definition.

LEMMA 3.18. *If $W_1 \cap W_2 \neq \varnothing$, $W_1 \subset U$, $W_2 \subset V$ and $\mathfrak{U}$ and $\mathfrak{V}$ interlock then $L(W_1, \mathfrak{U})$ and $L(W_2, \mathfrak{V})$ also interlock.*

LEMMA 3.19. *Suppose that $\mathfrak{U}$ and $\mathfrak{V}$ interlock. If $p \mid n$ and $\{p, m\}$ is divisible then $\mathfrak{U}_p$ and $\mathfrak{V}$ interlock.*

PROOF. Let $\mathfrak{U}_p = \{W_i \mid i = 0, \ldots, p-1\}$. By hypothesis $U_i \cap V_j \neq \varnothing$ iff $i \equiv j \bmod(\min\{n, m\})$.

Case I. $p \mid m$. If $W_i \cap V_j \neq \varnothing$ then for some $U_k \subset W_i$, $U_k \cap V_j \neq \varnothing$. This implies that $k \equiv j \bmod(\min\{n, m\})$, so $k \equiv j \bmod p$ because $p \mid m$ and $p \mid n$. Since $U_k \subset W_i$, $k \equiv i \bmod p$ and thus $i \equiv j \bmod p$. Conversely, if $0 \leqslant i < p$, $0 \leqslant j < m$ and $i \equiv j \bmod p$ then $U_{j \bmod n} \subset W_i$ and since $(j \bmod n) \equiv j \bmod(\min\{n, m\})$ and $\mathfrak{U}$ and $\mathfrak{V}$ interlock, $U_{j \bmod n} \cap V_j \neq \varnothing$ and so $W_i \cap V_j \neq \varnothing$.

Case II. $m \mid p$. Then $m \mid n$. Now, if $W_i \cap V_j \neq \varnothing$ then for some $U_k \subset W_i$, $U_k \cap V_j \neq \varnothing$. This implies that $k \equiv j \bmod m$. Since $U_k \subset W_i$, $k \equiv i \bmod p$ and so $k \equiv i \bmod m$. Therefore $i \equiv j \bmod m$. Conversely, if $0 \leqslant i < p$, $0 \leqslant j < m$ and $i \equiv j \bmod m$ then, since $\mathfrak{U}$ and $\mathfrak{V}$ interlock, $U_i \cap V_j \neq \varnothing$. Since $i \equiv i \bmod p$, $U_i \subset W_i$. Therefore $W_i \cap V_j \neq \varnothing$. $\square$

LEMMA 3.20. *Suppose $\mathfrak{U}$ and $\mathfrak{V}$ interlock. If $p \mid n$, $q \mid m$ and $\{p, q\}$ is divisible then $\mathfrak{U}_p$ and $\mathfrak{V}_q$ interlock.*

PROOF. Suppose $p \mid q$. Then $p \mid m$ so by the previous lemma $\mathfrak{U}_p$ and $\mathfrak{V}$ interlock. Apply the lemma again to $\mathfrak{U}_p$, $\mathfrak{V}$ and q and conclude that $\mathfrak{U}_p$ and $\mathfrak{V}_q$ interlock. The other case is similar. $\square$

COROLLARY 3.21. *Under the same hypothesis of the last lemma, if $W_1 \subset U$ and $W_2 \subset V$ then $L_p(W_1, \mathfrak{U})$ and $L_q(W_2, \mathfrak{V})$ interlock.*

LEMMA 3.22. *Suppose that $\mathfrak{U}$ and $\mathfrak{V}$ interlock and $m \mid n$. Then $L(U \cap V, \mathfrak{V}) = L_m(U \cap V, \mathfrak{U})$.*

PROOF. This is a consequence of the fact that

$$\begin{aligned}\pi^{-1}(U \cap V) \cap V_i &= \pi^{-1}(U \cap V) \cap \bigcup \{U_k \mid U_k \cap V_i \neq \varnothing\} \\ &= \pi^{-1}(U \cap V) \cap \bigcup \{U_k \mid k \equiv i \bmod m\} \\ &= \bigcup \{\pi^{-1}(U \cap V) \cap U_k \mid k \equiv i \bmod m\}. \quad \square\end{aligned}$$

COROLLARY 3.23. *Under the same hypothesis of the last lemma if $W \subset U \cap V$ then $L(W, \mathfrak{V}) = L_m(W, \mathfrak{U})$.*

In particular, when $\operatorname{ord} \mathfrak{U} = \operatorname{ord} \mathfrak{V}$, this result shows that no matter which one of $\mathfrak{U}$ and $\mathfrak{V}$ we use to lift W, we obtain the same lift, provided $\mathfrak{U}$ and $\mathfrak{V}$ interlock.

Now we want to present an equivalent definition of interlocking of two lifts.

THEOREM 3.24. *Suppose $\mathcal{U} \cap \mathcal{V} \neq \varnothing$. Then, $\mathcal{U}$ and $\mathcal{V}$ interlock iff each U_i intersects exactly one V_j or each V_i intersects exactly one U_j.*

PROOF. Suppose that $\mathcal{U}$ and $\mathcal{V}$ interlock and $m \mid n$. If U_i intersects both V_j and V_k then $j \equiv k \bmod m$ so $V_j = V_k$, since $0 \leqslant j, k \leqslant m-1$.

Suppose now that $\mathcal{U}$ and $\mathcal{V}$ do not interlock.

If $\{m, n\}$ is not divisible, let $x \in U_0 \cap V_0$. Then $h^m(x) \in h^m(U_0) \cap h^m(V_0)$. By Lemma 3.8, $h^m(U_0) \neq U_0$ and $h^m(V_0) = V_0$. Then V_0 intersects both $h^m(U_0)$ and U_0. Similarly $h^n(x) \in h^n(U_0) \cap h^n(V_0)$ so U_0 intersects both V_0 (cf. 3.15) and $h^n(V_0)$.

If $\{m, n\}$ is divisible, say $m \mid n$, and for some i, j, $U_i \cap V_j \neq \varnothing$ and $i \not\equiv j \bmod m$, then let $x \in U_i \cap V_j$. Then $h^m(x) \in h^m(U_i) \cap h^m(V_j)$. By Lemma 3.8, $h^m(U_i) \neq U_i$ and $h^m(V_j) = V_j$. Then V_j intersects both U_i and $h^m(U_i)$. Similarly $h^{-i}(x) \in h^{-i}(U_i) \cap h^{-i}(V_j)$ so U_0 intersects both V_0 (cf. 3.15) and $V_{j-i \bmod m}$.

In any case, we just proved that if $\mathcal{U}$ and $\mathcal{V}$ do not interlock then some U_i intersects more than one V_j and some V_i intersects more than one U_j. $\square$

LEMMA 3.25. *Let $C = \{C_i \mid i = 1, \ldots, n\}$ be an open finite cover of M_h. Suppose that for each $C_i \in C$ there exists a lift $L(C_i)$ of order n_i in such a way that whenever $C_i \cap C_j \neq \varnothing$ then $L(C_i)$ and $L(C_j)$ interlock. Then $\hat{C} = \bigcup \{L(C_i) \mid i = 1, \ldots, n\}$ is a finite open cover of M invariant under h. Furthermore* $\dim \hat{C} = \dim C$.

PROOF. Let $U \in \hat{C}$. Then $U \in L(C_i)$ for some i, which is a lift, so $h(U) \in L(C_i) \subset \hat{C}$. Therefore $\hat{C}$ is invariant. Let $W_j \in \hat{C}$ for $j = 1, \ldots, m$. Since all the elements of each $L(C_i)$ are disjoint, $\bigcap \{W_j \mid j = 1, \ldots, m\} \neq \varnothing$ iff $W_j \in L(C_{i_j})$ where all C_{i_j}'s are distinct. But, by Lemma 2.10, this happens iff $\bigcap \{C_{i_j} \mid j = 1, \ldots, m\} = \varnothing$. $\square$

LEMMA 3.26. *In addition to the hypothesis of the previous lemma, suppose that C is a tree-cover of M_h. Then $\mathfrak{N}(\hat{C})$ is the (C, σ)-bellows, where $\sigma(C_i) = n_i$. Furthermore, $\bar{h}$: $\mathfrak{N}(\hat{C}) \to \mathfrak{N}(\hat{C})$, the map induced by h, is an isomorphism which is also an isometry.*

PROOF. If C is a tree-cover, it is certainly a graph where the vertices are the elements of C and (C_1, C_2) is an edge iff $C_1 \cap C_2 \neq \varnothing$. Define $K = (C, \sigma)$ by $\sigma(C_i) = \operatorname{ord} L(C_i) = n_i$. By the interlocking condition in the hypothesis, K is a divisible $\#$-graph so it makes sense to talk about the K-bellows.

Let us verify that $\mathfrak{N}(\hat{C})$ is indeed the K-bellows. Since C is a tree-cover we can assume that if $i > 1$, C_i intersects exactly one C_j with $j < i$. Give any labeling (cf. Remark 3.2) to the elements of $L(C_1)$. Label the elements of $L(C_2)$ in such a way that the zero-labeled elements intersect. Continue labeling the elements of $L(C_i)$ consistently with the labeling of the elements of the unique $L(C_j)$, $j < i$, that it intersects. At the end, the zero-labeled elements of lifts that intersect will intersect.

Now recall that $\mathfrak{N}(\hat{C})$ is a finite simplical complex, a graph, whose set of vertices is $\bigcup \{L(C_i) \mid i = 1, \ldots, n\}$ where $L(C_i) = \{U_{i1}, U_{i2}, \ldots, U_{in_i}\}$. Observe next that (U_{il}, U_{jk}) is an edge of $\mathfrak{N}(\hat{C})$ iff $C_i \cap C_j \neq \varnothing$ and $i \equiv j \bmod(\min\{n_i, n_j\})$. This is true since $L(C_i)$ and $L(C_j)$ interlock. But this happens iff (C_i, C_j) is an edge of C and $i \equiv j \bmod(\min\{\sigma(C_i), \sigma(C_j)\})$. Then $\mathfrak{N}(\hat{C})$ is a K-bellows.

By the previous lemma, $\hat{C}$ is an invariant cover. The map $\bar{h}$ is defined by the linear extension of the correspondence $U_{il} \to h(U_{il})$. Since h is a homeomorphism, $\bar{h}$ is an isomorphism. By properly choosing the metric in the nerve (Kaul [**K**]), $\bar{h}$ becomes an isometry. □

The necessity of C being a tree-cover is illustrated by the following

EXAMPLE 3.27. Let M be the space shown in Figure 3.2(a) with h being the period-2 homeomorphism that maps parts labeled with "1" to parts labeled with "2". Figure 3.2(b) shows M_h.

Consider the cover C of Figure 3.2(d). C is not a tree and lifts to the graph $\hat{C}$ shown in Figure 3.2(c). The number graph (C, σ) (Figure 3.2(e)) has its bellows displayed in Figure 3.2(f), (g). But $\mathfrak{N}(\hat{C})$ (Figure 3.2(h)) is not the (C, σ)-bellows!

The difficulty arises in trying to label the elements of the lifts of each C_i in a consistent way as illustrated in Figure 3.2(i).

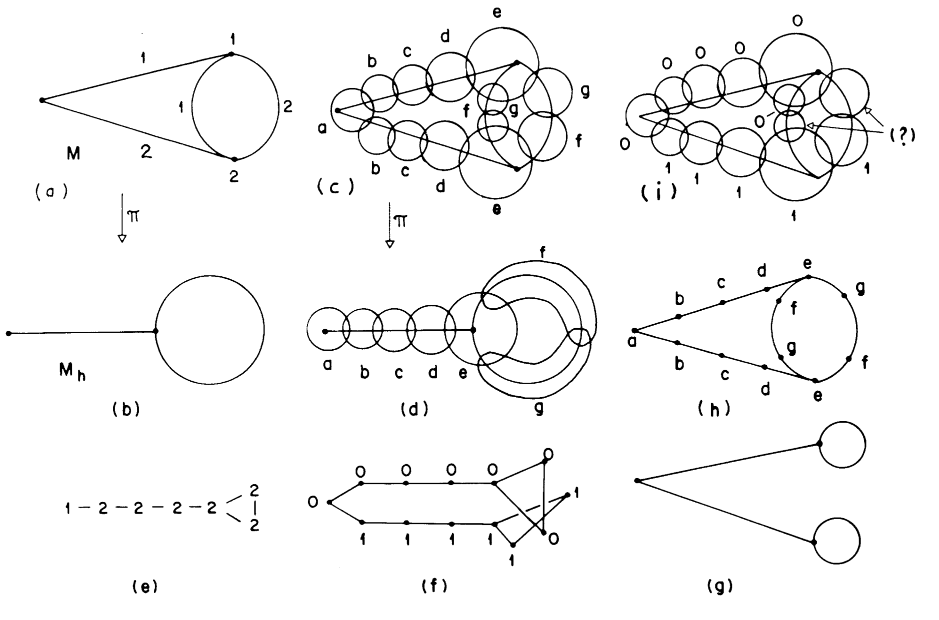


FIGURE 3.2

4. Round and spread covers. In this section we introduce the concept and prove the existence of a very useful type of invariant cover that allows us to construct invariant one-dimensional covers having bellows as nerves.

The main results of this section are Theorems 4.13, 4.17, 4.18, 4.22 and 4.24, and Corollary 4.20.

Throughout this section we assume that h: $M \to M$ is a period-n homeomorphism of a continuum M. We will denote the orbit space of h by M_h and π: $M \to M_h$ will be the projection map.

Since M is metric and the set $\{h, h^2, h^3, \ldots, h^n\}$ is an equicontinuous group of homeomorphisms we can endow M with a metric d: $M \times M \to R$ with respect to which h is an isometry [**A**, p. 604]. Denote by $B_\delta(x)$ the set $\{y \in M \mid d(x, y) < \delta\}$, i.e., the ball with center at x and radius δ. Since h is an isometry $h^s(B_\delta(x)) = B_\delta(h^s(x))$. Denote this set by $B_\delta^s(x)$. Observe that if $h(x) = x$ then $B_\delta^s(x) = B_\delta^t(x)$ for all s and t.

LEMMA 4.1. *For every $x \in M$ there exists a $\gamma > 0$ such that the set $\{\overline{B_\delta^s(x)} \mid s = 0,\ldots,\text{ord } x - 1\}$ is pairwise disjoint.*

PROOF. Let $\gamma < \alpha/2$ where

$$\alpha = \min\{d(h^s(x), h^t(x)) \mid 0 \leq s \neq t \leq \text{ord } x - 1\}. \quad \square$$

DEFINITION 4.2. A number γ with the above property is called a *splitting number for x*.

Observe that if γ is a splitting number for x so is any δ such that $0 < \delta < \gamma$. If $h(x) = x$ then any number is a splitting number for x.

DEFINITION 4.3. Define $T_\delta[x] = \bigcup\{B_\delta^s(x) \mid s = 0,\ldots,(\text{ord } x) - 1\}$ and $S_\delta[x] = \{B_\delta^s(x) \mid s = 0,\ldots,(\text{ord } x) - 1\}$.

Note that $T_\delta[h^s(x)] = T_\delta[x] = h^s(T_\delta[x])$ for all s.

LEMMA 4.4. *If δ is a splitting number for x, then $S_\delta[x]$ is an* (ord x)-*lift of* $\pi(T_\delta[x])$.

PROOF. By the choice of δ, Definition 3.1(i) is satisfied whereas Definition 3.1(ii) is a consequence of

$$\begin{aligned} h^s(B_\delta^i(x)) &= h^s(B_\delta(h^i(x))) = B_\delta(h^{s+i}(x)) \\ &= B_\delta(h^{s+i \bmod n}(x)) = B_\delta^{s+i \bmod n}(x). \end{aligned}$$

The observation that if $y \in T_\delta[x]$ then $O(y) \subset T_\delta[x]$ proves that $\pi^{-1}\pi(T_\delta[x]) = T_\delta[x] = \bigcup S_\delta[x]$ or Definition 3.1(iii). $\square$

REMARK 4.5. Observe that $T_\delta[x]$ is the 1-consolidation of $S_\delta[x]$.

DEFINITION 4.6. Let $\mathcal{C}$ be a finite open cover of M. We say that $\mathcal{C}$ is a *round cover of M* iff $\mathcal{C} = \{T_{\gamma_i}[a_i] \mid i = 1,\ldots,\mu\}$ where $a_i \in M$ and γ_i is a splitting number for a_i. The *size of* $\mathcal{C}$ is defined by size $\mathcal{C} = \max\{2\gamma_i \mid i = 1,\ldots,\mu\}$. The *set of centers of* $\mathcal{C}$ is the set $i(\mathcal{C}) = \{a_i \mid i = 1,\ldots,\mu\}$.

REMARK 4.7. Observe that any of a_i, $h(a_i),\ldots,h^{(\text{ord } a_i)-1}(a_i)$ can serve as the center of $T[a_i]$. Therefore, if $T[a_i] \cap T[a_j] \neq \varnothing$, it is not necessarily true that $B(a_i) \cap B(a_j) \neq \varnothing$. However, once the centers are specified they will remain fixed unless otherwise stated.

Sometimes we will omit the radius γ_i in order to simplify the notation.

It is easy to get round covers of size less then ε. Simply consider $\{T_{\gamma_x}[x] \mid x \in M\}$ where $\gamma_x < \varepsilon$ is a splitting number for x and use the compactness of M to extract a finite subcover.

DEFINITION 4.8. If $\mathcal{C} = \{T_{\gamma_i}[a_i] \mid i = 1,\ldots,\mu\}$ is a round cover of M then the *Shred of* $\mathcal{C}$ is the finite open cover Shr $\mathcal{C} = \bigcup\{S_{\gamma_i}[a_i] \mid i = 1,\ldots,\mu\}$.

Informally, Shr $\mathcal{C}$ is the lift of $\pi(\mathcal{C})$ formed by the lifts of the elements of $\pi(\mathcal{C})$, whereas $\mathcal{C}$ is the "1-consolidation" of Shr $\mathcal{C}$.

Notation 4.9. From now on, any time we say a_i and a_j interlock, we actually mean $S[a_i]$ and $S[a_j]$ interlock.

DEFINITION 4.10. Let $\mathcal{C}$ be a round cover of M. We say that $\mathcal{C}$ is a *spread cover of* M iff whenever $a_i, a_j \in I(\mathcal{C})$ are such that $T[a_i] \cap T[a_j] \neq \varnothing$ then a_i and a_j interlock. We will also say that a_i *misintersects* a_j iff $T[a_i] \cap T[a_j] \neq \varnothing$ and a_i and a_j do not interlock.

Let us now take a look at Figure 4.1 to illustrate the concepts just introduced.

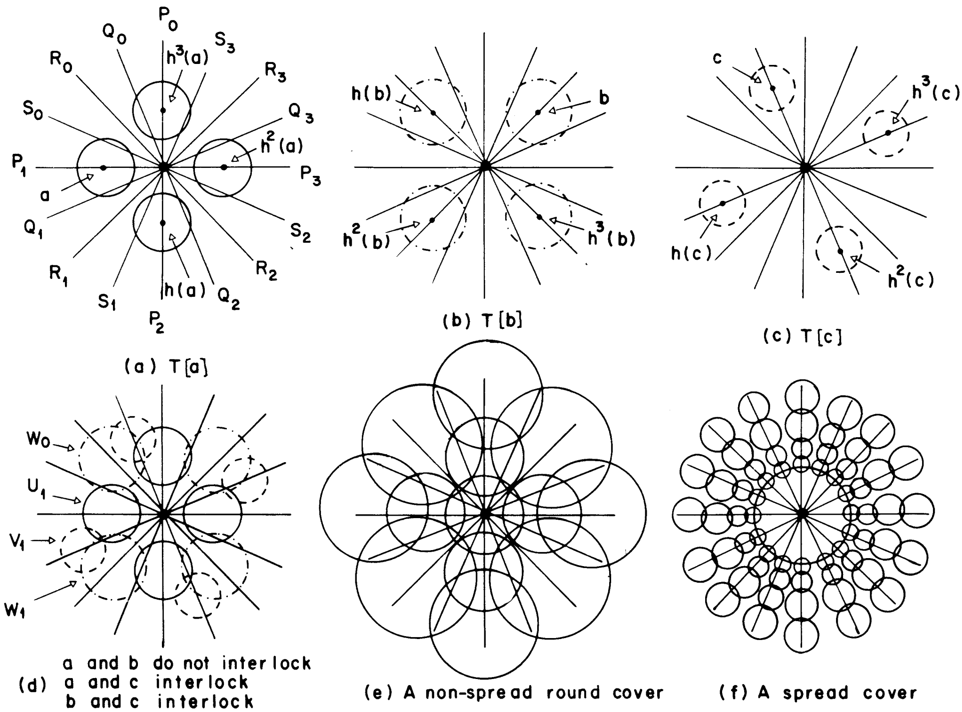


FIGURE 4.1

Let M be the space in the plane consisting of the tree formed by 16 lines joined at one endpoint and with the angle $\pi/8$ between any two of them. Let $h\colon M \to M$ be the homeomorphism defined by a 90° rotation in the positive direction.

Then, in Figure 4.1(a), the lines P_i, Q_i, R_i, S_i, $0 \leqslant i \leqslant 3$, are mapped by h linearly to the lines P_j, Q_j, R_j, S_j, respectively, where $j \equiv i + 1 \bmod 4$. Observe that under the usual Euclidean metric, h is an isometry.

Let $a \in P_1$ with ord $a = 4$ and consider the ball $B_\delta(a)$ as shown in Figure 4.1(a). Since the interates of this ball are all disjoint, δ is a splitting number for a. Thus $T[a] = \bigcup\{B_\delta^i(a) \mid i = 0,\ldots,3\}$ is a lift of $\pi(T[a])$ of order 4.

Consider the similar situation for $b \in R_3$ and $c \in Q_0$ displayed in Figure 4.1(b) and (c). Let us put the three cases together considering $T[a]$, $T[b]$ and $T[c]$ as lifts.

Using Remark 3.2 and Standing Notation 3.15 we can define

$$T[a] = \bigcup \{U_i \mid i = 0,\ldots,3\} \quad \text{where } U_i = B^{i+3 \bmod 4}(a),$$

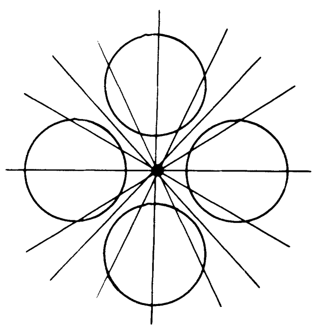
$$T[b] = \bigcup \{W_i \mid i = 0,\ldots,3\} \quad \text{where } W_i = B^{i+1 \bmod 4}(b),$$

$$T[c] = \bigcup \{V_i \mid i = 0,\ldots,3\} \quad \text{where } V_i = B^{i}(c).$$

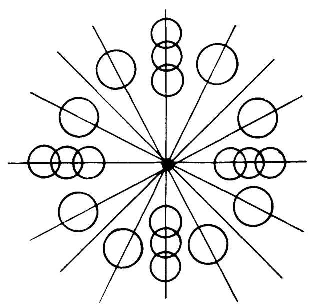
A glance at Definition 3.4 and Figure 4.1(d) is enough to realize that a and b do not interlock, whereas a and c, and b and c, do. In Figure 4.1(e) and (f) we can compare two round covers which are nonspread and spread, respectively.

We want to prove now the existence of spread covers of arbitrarily small mesh. We will do this by indicating how to obtain a spread cover out of a round cover of convenient mesh.

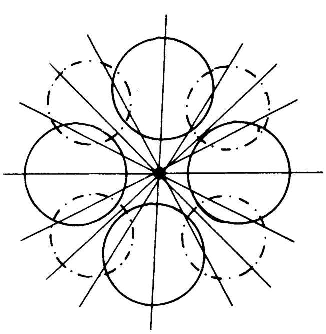
The method, contained in the proof of the next lemma, consists in replacing (breaking up) all those $T[a]$'s that produce misintersections by (into) a round refinement $\{T[\alpha_k] \mid k = 1,\ldots,\mu\}$ with $T[\alpha_k] = \cup\{Z_{ki}\}$, which does not produce any misintersections (see Figure 4.2).



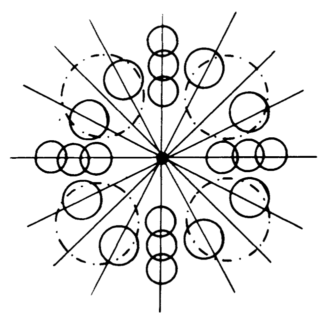
(a) T[a]



(c) The break-up



(b) The problem



(d) The solution

FIGURE 4.2

LEMMA 4.11. *For any round cover $\mathcal{C} = \{T[a_i] \mid i = 1,\ldots,\mu\}$ of M and any $a_p \in I(\mathcal{C})$ there exist $\alpha_k \in M$, $k = 1,\ldots,\mu_1$, such that $T[a_p] \subset \cup\{T[a_k] \mid k = 1,\ldots,\mu_1\}$ and no α_j misintersects any element of $\{a_i \mid i \neq p\} \cup \{a_k \mid k \neq j\}$.*

PROOF. If $\operatorname{ord} a_i = 1$ for all $i = 1,\ldots,\mu$ then there are no misintersections whatsoever ($\mathcal{C}$ is therefore a spread cover) and the lemma is trivially true. Therefore, let us assume that at least for some i, $\operatorname{ord} a_i > 1$.

We will use finite induction on $\operatorname{ord} a_p$ over the #-graph Γ_h (cf. Definition 2.11) partially ordered by "$i < j$ iff $j \mid i$".

Let $\delta = \min\{\operatorname{dist}(B^s(a_i), B^t(a_i)) \mid \operatorname{ord} a_i > 1;\ 1 \leqslant s < t < \operatorname{ord} a_i;\ i = 1,\ldots,\mu\}$. Since $\operatorname{ord} a_i > 1$, for some i, δ is positive. Cover $\overline{B(a_p)}$ with $\{B_{\gamma_j}(\alpha_j) \mid j = 1,\ldots,\mu_p\}$ where $\alpha_j \in \overline{B(a_p)}$ and γ_j is a splitting number for α_j such that $\gamma_j < \delta/4$. Replace now $T[a_p]$ by $\{T_{\gamma_j}[\alpha_j] \mid j = 1,\ldots,\mu_p\}$.

Now suppose some α_j misintersects some a_i, $i \neq p$. This means that α_j and a_i do not interlock so Theorem 3.24 implies that some $B^s(\alpha_j)$ intersects both $B^{s_1}(a_i)$ and $B^{s_2}(a_i)$ for some s, s_1, s_2. But then (Figure 4.3(a))

$$d(B^{s_1}(a_i), B^{s_2}(a_i)) < \operatorname{diam} B^s(\alpha_j) < 2\gamma(j) < \delta/2 < \delta$$

which is impossible. Then no α_j misintersects any a_i, $i \neq p$.

But it may be that there are some misintersections among the α_j's. Let us now use the induction argument to get rid of those newly-generated misintersections.

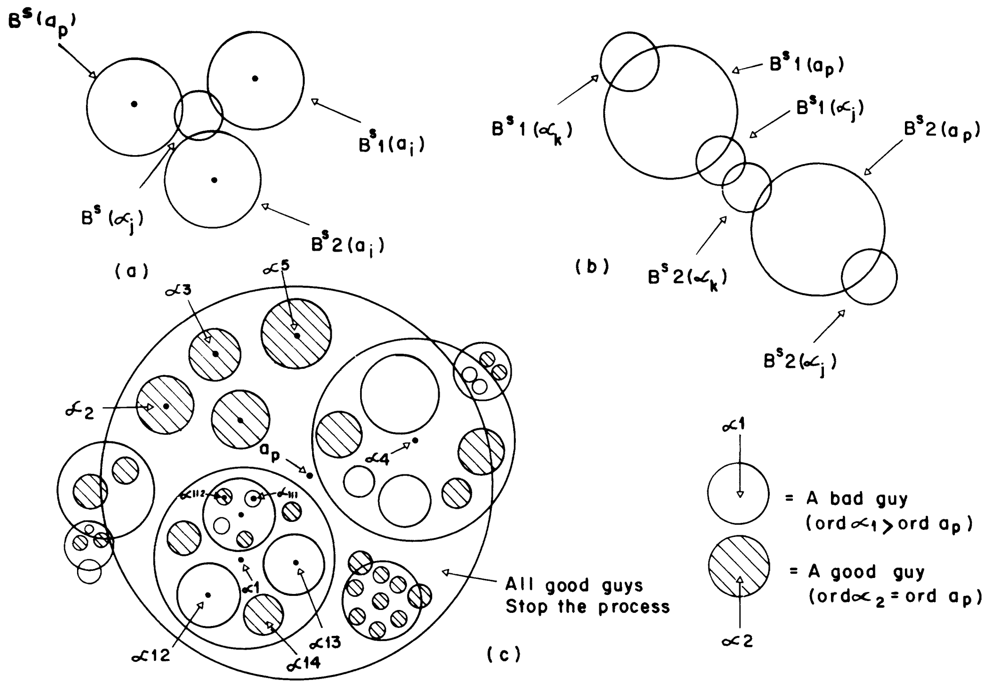


FIGURE 4.3

Initial step. Suppose $\operatorname{ord} a_p$ is maximal. By Lemma 3.9 $\operatorname{ord} a_p \mid \operatorname{ord} \alpha_j$ for all j, since $\alpha_j \in \overline{B(a_p)}$. But since $\operatorname{ord} a_p$ is maximal, $\operatorname{ord} a_p = \operatorname{ord} \alpha_j$ for all j. This means that for all s and all j, $B^s(\alpha_j)$ intersects only $B^s(a_p)$, otherwise $\operatorname{dist}(B^s(a_p), B^{s_1}(a_p)) < \delta$ for some s_1.

Now suppose that α_j misintersects some α_k. Theorem 3.24 implies that $B^{s_1}(\alpha_j)$ intersects $B^{s_2}(\alpha_k)$ for some $s_1 \neq s_2$. But then

$$\begin{aligned} d(B^{s_1}(a_p), B^{s_2}(a_p)) &\leqslant \operatorname{diam} B^{s_1}(\alpha_j) + \operatorname{diam} B^{s_2}(\alpha_k) \\ &\leqslant 2\gamma(j) + 2\gamma(k) < \delta/2 + \delta/2 = \delta \end{aligned}$$

which is impossible (Figure 4.3(b)). Then no α_j misintersects any α_k and the argument is completed.

Inductive step (see Figure 4.3(c)). Suppose now that ord a_p is not maximal and that the lemma is true for any round cover $\mathfrak{B}$ and any $b \in I(\mathfrak{B})$ such that ord $a_p \mid$ ord b and ord $a_p \neq$ ord b.

Since $\alpha_j \in \overline{B(a_p)}$, Lemma 3.9 tells us that ord $a_p \mid$ ord α_j. Suppose first that ord $\alpha_j =$ ord a_p and that α_j misintersects some α_k. Again, Theorem 3.24 implies that for some s, s_1, s_2, $B^s(\alpha_k)$ meets both $B^{s_1}(\alpha_j)$ and $B^{s_2}(\alpha_j)$ making the distance between $B^{s_1}(a_p)$ and $B^{s_2}(a_p)$ less than δ, a contradiction. Then we do not have to worry about a_j's with ord $a_j =$ ord a_p, so if ord $a_j =$ ord a_p for all j then we are done.

The only problems now are the α_j's such that ord $a_p \mid$ ord α_j and ord $a_p \neq$ ord α_j. Let us rename them as α_i, $i = 1, \ldots, l$. Apply now the induction hypothesis to the round cover $\{T[\alpha_j] \mid j = 1, \ldots, l\} \cup \{T[a_i] \mid i \neq p\}$ and α_1 to obtain a new round cover $\{T[\alpha_{1k}]\} \cup \{T[\alpha_j] \mid j \neq 1\} \cup \{T[a_i] \mid i \neq p\}$ where no α_{1k} misintersects any other element. Do the same now for this cover and α_2 to obtain another round cover $\{T[\alpha_{1k}]\} \cup \{T[\alpha_{2k}]\} \cup \{T[\alpha_j] \mid j \neq 1, 2\} \cup \{T[a_i] \mid i \neq p\}$ where no α_{1k} and no α_{2k} misintersect any other element.

Proceed this way with all the α_i, $i = 1, \ldots, l$. The final cover so obtained satisfies the conditions of the statement of the lemma. □

REMARK 4.12. Observe that a spread cover is one in which no element misintersects any other element. With this in mind we proceed to prove the main result of this section.

THEOREM 4.13. *Given $\varepsilon > 0$, there exists a spread cover $\mathfrak{A}$ such that* size $\mathfrak{A} < \varepsilon$.

PROOF. Start with any round cover $\mathfrak{A} = \{T[a_i] \mid i = 1, \ldots, \mu\}$ of size less than ε.

Consider $a_1 \in I(\mathfrak{A})$. Use the previous lemma to break up $T[a_1]$ into $\{T_{\gamma_{1k}}[\alpha_{1k}] \mid k = 1, \ldots, \mu_1\}$ where $\gamma_{1k} < \varepsilon/2$. We obtain a new round cover $\{T[\alpha_{1k}] \mid k = 1, \ldots, \mu_1\} \cup \{T[a_i] \mid i = 2, \ldots, \mu\}$ in which no α_{1k} misintersects any other element.

Now look at α_2 and use the last lemma again to obtain another round cover

$$\{T[\alpha_{1k}] \mid k = 1, \ldots, \mu_1\} \cup \{T[\alpha_{2k}] \mid k = 1, \ldots, \mu_2\} \cup \{T[a_i] \mid i = 3, \ldots, \mu\}$$

of size less than ε and such that no element misintersects any other element.

Continuing in this fashion we end up with a round cover $\mathfrak{A} = \{T[\alpha_{ik}] \mid i = 1, \ldots, \mu; k = 1, \ldots, \mu_i\}$ of size less than ε and such that no element misintersects any other element.

$\mathfrak{A}$ is therefore a spread cover of size less than ε. □

REMARK 4.14. Suppose that $U \subset M_h$ is such that $U < \pi(\mathfrak{A})$, where $\mathfrak{A}$ is a spread cover of M. Consider $H = \{a_i \in I(\mathfrak{A}) \mid U \subset \pi(T[a_i])\}$. Observe that a_i, $a_j \in H$ implies that $T[a_i] \cap T[a_j] \neq \varnothing$ and since $\mathfrak{A}$ is spread, a_i and a_j interlock. Therefore the set $\{\text{ord } a_i \mid a_i \in H\}$ is pairwise divisible. Furthermore, by Corollary 3.23, $\{L(U, S[a_i]) \mid a_i \in H\}$ is such that out of every pair of lifts, one is a consolidation of the other. That is, it is a totally ordered set under refinement, using Lemma 3.6. Therefore there is a maximal element.

This allows us to have, without ambiguity, the following

DEFINITION 4.15. Let $\mathcal{C}$ be a spread cover of M and $U < \pi(\mathcal{C})$ where U is an open set. Define the *lift of U induced by $\mathcal{C}$*, $L(U, \mathcal{C})$, by $L(U, \mathcal{C}) = L(U, S[a_i])$ where $\operatorname{ord} a_i = \max\{\operatorname{ord} a_j \mid U \subset \pi(T[a_j])\}$. If $m \mid \operatorname{ord} a_i$, $L_m(U, \mathcal{C})$ denotes the m-consolidation of $L(U, \mathcal{C})$. If $C = \{C_i\}$ is a finite open cover of M_h such that $C < \pi(\mathcal{C})$, then the *lift of C induced by $\mathcal{C}$*, $L(C, \mathcal{C})$, is defined to be the cover $L(C, \mathcal{C}) = \bigcup\{L(C_i, \mathcal{C}) \mid C_i \in C\}$.

REMARK 4.16. It follows from this definition that if $U \subset \pi(T[a])$, $a \in I(\mathcal{C})$, then $L(U, S[a]) = L_{\operatorname{ord} a}(U, \mathcal{C})$ and that, by Lemma 3.7, if $p \mid \operatorname{ord} a$ then, $L_p(U, S[a]) = L_p(U, \mathcal{C})$.

THEOREM 4.17. *Let $\mathcal{C}$ be a spread cover of size less than ε and $C < \pi(\mathcal{C})$ a finite open cover of M_h. Then $L(C, \mathcal{C})$ is a finite open cover of M, invariant under h, of mesh less than ε. Furthermore*, $\dim L(C, \mathcal{C}) = \dim C$.

PROOF. Let $U \in L(C, \mathcal{C})$. Then $U < S[a]$ for some $a \in I(\mathcal{C})$ so $U \subset B^s_\gamma(a)$ for some s. Therefore $\operatorname{diam} U < \operatorname{diam} B^s_\gamma(a) < 2\gamma \leqslant \operatorname{size} \mathcal{C} < \varepsilon$ and so mesh $L(C, \mathcal{C}) < \varepsilon$.

Now let U, $V \in C$, such that $U \cap V \neq \varnothing$. Suppose that a, $b \in I(\mathcal{C})$ are centers of maximal orders such that $U \subset \pi(T[a])$ and $V \subset \pi(T[b])$, respectively. Then $T[a] \cap T[b] \neq \varnothing$ and since $\mathcal{C}$ is spread, a and b interlock. Then $L(U, \mathcal{C})$ and $L(V, \mathcal{C})$ or, equivalently, $L(U, S[a])$ and $L(V, S[b])$ interlock, following Lemma 3.18.

All we have to do now is to apply Lemma 3.25 to conclude the proof. □

THEOREM 4.18. *Let $\mathcal{C}$ be a spread cover of M. If $C < \pi(\mathcal{C})$ is a tree-cover of M_h, then $\mathfrak{N}(L(C, \mathcal{C}))$ is a bellows. Furthermore $\bar{h}$: $\mathfrak{N}(L(C, \mathcal{C})) \to \mathfrak{N}(L(C, \mathcal{C}))$, the map induced by h, is an isomorphism which is also an isometry.*

PROOF. Use the previous lemma in conjunction with Lemma 3.26. □

REMARK 4.19. We can use again Example 3.27 to show the necessity of C being a tree-cover, by noticing that the covers used in that example were a spread cover and its projection.

COROLLARY 4.20. *Let h: $M \to M$ be a periodic homeomorphism such that M_h is tree-like. Then, for every $\varepsilon > 0$, there exists a finite open cover C of M of mesh less than ε, invariant under h and such that $\mathfrak{N}(C)$ is a* 1*-dimensional bellows.*

PROOF. First, get a spread cover $\mathcal{C}$ of size less than ε. Then use the Lebesgue number of $\pi(\mathcal{C})$ and the fact that M_h is tree-like to find a tree-cover C of M_h such that $C < \pi(\mathcal{C})$. Finally, apply Theorems 4.17 and 4.18. □

Now we want to generalize Theorems 4.17 and 4.18 in the following way.

DEFINITION 4.21. Let $\mathcal{C} = \{T[a_i] \mid i = 1,\ldots,\mu\}$ be a spread cover of M and $C = \{C_i \mid i = 1,\ldots,\lambda\}$ a tree-cover of M_h such that $C < \pi(\mathcal{C})$. Let σ: $C \to N^+$ be a divisible #-graph such that $\sigma(C_i) \mid \operatorname{ord} L(C_i, \mathcal{C})$ for all $C_i \in C$. Define the *σ-lift of C induced by $\mathcal{C}$*, $L_\sigma(C, \mathcal{C})$, by $L_\sigma(C, \mathcal{C}) = \bigcup\{L_{\sigma(C_i)}(C_i, \mathcal{C}) \mid C_i \in C\}$.

THEOREM 4.22. *Let $\mathcal{C}$ be a spread cover of M, $C < \pi(\mathcal{C})$ a finite open cover of M_h and (C, σ) a divisible #-graph such that $\sigma(C_i) \mid \operatorname{ord} L(C_i, \mathcal{C})$. Then $L_\sigma(C, \mathcal{C})$ is a finite open cover of M invariant under h and* $\dim L_\sigma(C, \mathcal{C}) = \dim C$.

PROOF. Since (C, σ) is divisible, if $C_i, C_j \in C$ are such that $C_i \cap C_j \neq \varnothing$ then $\{\sigma(i), \sigma(j)\}$ is divisible. Then, by Corollary 3.21, $L_{\sigma(C_i)}(C_i, \mathcal{C})$ and $L_{\sigma(C_j)}(C_j, \mathcal{C})$ interlock. Now apply Lemma 3.25. $\square$

REMARK 4.23. Notice that in this case size $\mathcal{C} < \varepsilon$ does not necessarily imply mesh $L_\sigma(C, \mathcal{C}) < \varepsilon$ since some elements of $L_\sigma(C, \mathcal{C})$ may be unions of elements of $\operatorname{Shr}\mathcal{C}$ too far apart (recall that if $\sigma(C_i) \neq \operatorname{ord} L(C_i, \mathcal{C})$, then we are performing consolidations).

The next theorem is the analogue of Theorem 4.18 and follows from Lemmas 3.21 and 3.26.

THEOREM 4.24. *Let $\mathcal{C}$ be a spread cover of M, $C < \pi(\mathcal{C})$ a tree-cover of M_h and (C, σ) a divisible #-graph such that $\sigma(C_i) \mid \operatorname{ord} L(C_i, \mathcal{C})$. Then $\mathfrak{N}(L_\sigma(C, \mathcal{C}))$ is the (C, σ)-bellows and $\bar{h}$, the map induced by h on $\mathfrak{N}(L_\sigma(C, \mathcal{C}))$, is an isomorphism which is also an isometry.*

Observe that Theorems 4.17 and 4.18 are particular cases of the two previous theorems, namely for the case when $\sigma(C_i) = \operatorname{ord} L(C_i, \mathcal{C})$. We can then refer in general to $L_\sigma(C, \mathcal{C})$.

LEMMA 4.25. *Let $\mathcal{C}$ be a spread cover of M, $C < \pi(\mathcal{C})$ a tree-cover of M_h, (C, σ) a divisible #-graph such that $\sigma(C_i) \mid \operatorname{ord} L(C_i, \mathcal{C})$ and $\varepsilon > 0$. Then $N = \mathfrak{N}(L_\sigma(C, \mathcal{C}))$ has a simplical subdivision N' of diameter less than ε that is invariant under $\bar{h}$. That is, if p is a vertex of N', then $\bar{h}(p)$ also is.*

PROOF. Recall that the vertices of N are of the form U_{Ei}, where $E \in C$ and $0 \leqslant i < \sigma(E)$.

Suppose that (U_{Ei}, U_{Dj}) is an edge of N so, say, $\sigma(E) \mid \sigma(D)$ and $i \equiv j \bmod \sigma(E)$. Then $h^j(U_{E0}) = U_{E(j \bmod \Sigma(E))} = U_{Ei}$ and $h^j(U_{D0}) = U_{Dj}$. Then $\bar{h}^j((U_{E0}, U_{D0})) = (E_{Ei}, U_{Dj})$, since $\bar{h}$ is a linear extension of h.

Now consider the subgraphs $\bar{N}$ of N generated by all those vertices of N labeled with "0". By the previous discussion, $\bar{N} \cup h(\bar{N}) \cup \cdots \cup h^n(\bar{N}) = N$. Find a subdivision $\bar{N}'$ of $\bar{N}$ of diameter less than ε. Since, by Theorem 4.24 $\bar{h}$ is an isometry, $N' = \bar{N}' \cup h(\bar{N}') \cup \cdots \cup h^n(\bar{N}')$ is a simplical subdivision of N of diameter less than ε. Clearly, N' is invariant under $\bar{h}$ (see Figure 4.4). $\square$

5. Inducing periodic homeomorphisms. In this section we investigate the conditions under which a periodic homeomorphism of a continuum onto itself can be induced as the limit of an inverse sequence of homeomorphisms.

We also investigate the kinds of spaces that the resulting inverse sequences are composed of.

Throughout this section M denotes a metric continuum, and if h is a periodic homeomorphism of M onto itself, M_h denotes the orbit space of M under h and π: $M \to M_h$ the projection map.

Our main results for this section are Theorems 5.3, 5.13 and 5.15, and Example 5.17.

We start the sections with a basic definition.

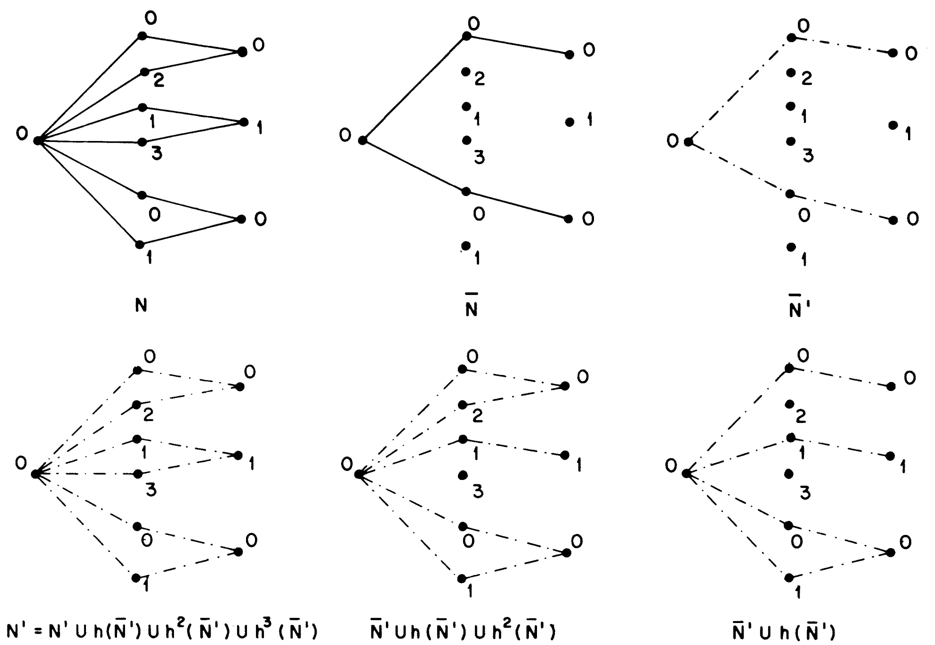


FIGURE 4.4

DEFINITION 5.1. A homeomorphism h: $M \to M$ is *inducible* iff there exists a commutative diagram

$$\begin{array}{ccccccccc}
P_1 & \overset{\alpha_1}{\leftarrow} & P_2 & \overset{\alpha_2}{\leftarrow} & P_3 & \overset{\alpha_3}{\leftarrow} & \cdots & P & \overset{\alpha}{\rightarrow} & M \\
f_1 \downarrow & & f_2 \downarrow & & f_3 \downarrow & & & f \downarrow & & \downarrow h \\
Q_1 & \overset{\beta_1}{\leftarrow} & Q_2 & \overset{\beta_2}{\leftarrow} & Q_3 & \overset{\beta_3}{\leftarrow} & \cdots & Q & \overset{\beta}{\rightarrow} & M
\end{array}$$

where $P = \varprojlim \{P_i, \alpha_i\}$, $Q = \varprojlim \{Q_i, \beta_i\}$, P_i and Q_i are finite simplicial complexes, α and β are homeomorphisms, f_i are simplical homeomorphisms, and f is the homeomorphism induced by $\{f_i\}$. If in addition $\alpha = \beta$ (and necessarily $P_i = Q_i$ and $P = Q$) h is said to be *strongly inducible*.

Kaul [**K**] proved that any homeomorphism of compact metric spaces is inducible. This was accomplished by constructing a commutative diagram

$$\begin{array}{cccccccc}
\mathfrak{N}(\mathfrak{U}_1) & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) & \overset{\pi_2}{\leftarrow} & \cdots & P & \overset{\alpha}{\rightarrow} & M \\
\downarrow f_1 & & \downarrow f_2 & & & f \downarrow & & \downarrow h \\
\mathfrak{N}(\mathcal{V}_1) & \overset{\phi_1}{\leftarrow} & \mathfrak{N}(\mathcal{V}_2) & \overset{\phi_2}{\leftarrow} & \cdots & Q & \overset{\beta}{\rightarrow} & M
\end{array}$$

where $\mathfrak{U}_i$ are finite open covers of M of mesh less than $1/2^i$, $\mathcal{V}_i = h(\mathfrak{U}_i)$, $\mathfrak{N}(\mathfrak{U})$ denotes the nerve of $\mathfrak{U}$ and f_i are the simplicial maps induced by h.

DEFINITION 5.2. A homeomorphism h: $M \to M$ is *B-inducible* iff:

(i) h is strongly inducible.

(ii) For all i, P_i in Definition 5.1 is a bellows.

Using the material of §§3 and 4 which is an elaboration of the techniques developed by Fugate and McLean **[FM]** and referring to the work by Kaul **[K]**, we proceed to prove the following

THEOREM 5.3. *Let h: $M \to M$ be a periodic homeomorphism of a metric continuum M such that M_h is tree-like. Then h is B-inducible.*

PROOF. We will follow very closely the proof of Theorem 4 in **[K]**. For definitions of the concepts used in this proof refer to **[K]**.

Use the fact that M_h is tree-like and Corollary 4.20 to find an invariant finite open cover $\mathcal{U}_1$ of M of mesh less than $\frac{1}{2}$ such that $\mathcal{N}(\mathcal{U}_1)$ is a bellows.

Let f_1: $\mathcal{N}(\mathcal{U}_1) \to \mathcal{N}(\mathcal{U}_1)$ be the homeomorphisms induced by h. Since h is an isometry, we can construct a barycentric map **[K**, p. 297], α_1: $M \to \mathcal{N}(\mathcal{U}_1)$ such that $f_1\alpha_1 = \alpha_1 h$.

Use Lemma 4.25 to find a simplicial subdivison $\mathcal{N}(\mathcal{U}_1)'$ of $\mathcal{N}(\mathcal{U}_1)$ of mesh less than $\frac{1}{2}$, invariant under f_1. Then

$$h\big(\alpha_1^{-1}(\mathrm{st}(p))\big) = \alpha^{-1}\big(\mathrm{st}(f_1(p))\big)$$

where p is any vertex of $\mathcal{N}(\mathcal{U}_1)'$ and the open stars are taken with respect to $\mathcal{N}(\mathcal{U}_1)'$.

Let $\mathcal{U}_{1.5}$ be the auxiliary cover **[K**, (1.1)], of M with respect to α_1 and $\mathcal{N}(\mathcal{U}_1)'$. $\mathcal{U}_{1.5}$ is invariant under h so $f_{1.5}$: $\mathcal{N}(\mathcal{U}_{1.5}) \to \mathcal{N}(\mathcal{U}_{1.5})$, the map induced by h, is an isomorphism. Furthermore, the diagram

$$\begin{array}{ccc} \mathcal{N}(\mathcal{U}_1)' & \overset{i_1}{\leftarrow} & \mathcal{N}(\mathcal{U}_{1.5}) \\ \downarrow f_1 & & \downarrow f_{1.5} \\ \mathcal{N}(\mathcal{U}_1)' & \overset{i_1}{\leftarrow} & \mathcal{N}(\mathcal{U}_{1.5}) \end{array}$$

commutes, where i_1 is a simplicial isomorphism **[K**, Lemma 2.1].

Again use Corollary 4.20 to find an invariant finite open cover $\mathcal{U}_2$ of M of mesh less than $1/2^2$ and such that $\overline{\mathcal{U}_2} < \mathcal{U}_{1.5}$ and $\mathcal{N}(\mathcal{U}_2)$ is a bellows. Furthermore, if f_2: $\mathcal{N}(\mathcal{U}_2) \to \mathcal{N}(\mathcal{U}_2)$ is the homeomorphism induced by h, then, since $\mathcal{N}(\mathcal{U}_2)$ is a bellows, we can construct a projection map **[K**, (1.3)] π_1': $\mathcal{N}(\mathcal{U}_2) \to \mathcal{N}(\mathcal{U}_{1.5})$ in such a way that the diagram

$$\begin{array}{ccc} \mathcal{N}(\mathcal{U}_{1.5}) & \overset{\pi_1'}{\leftarrow} & \mathcal{N}(\mathcal{U}_2) \\ \downarrow f_{1.5} & & \downarrow f_2 \\ \mathcal{N}(\mathcal{U}_{1.5}) & \overset{\pi_1'}{\leftarrow} & \mathcal{N}(\mathcal{U}_2) \end{array}$$

commutes.

Define $\pi_1 = i_1 \circ \pi_1'$ and put together the two previous diagrams to obtain the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{N}(\mathfrak{U}_1)' & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) \\ \downarrow f_1 & & \downarrow f_2 \\ \mathfrak{N}(\mathfrak{U}_1)' & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) \end{array}$$

Since $\mathfrak{N}(\mathfrak{U}_1)'$, as a space, is the same as $\mathfrak{N}(\mathfrak{U}_1)$ we actually obtained a commutative diagram:

$$\begin{array}{ccc} \mathfrak{N}(\mathfrak{U}_1) & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) \\ \downarrow f_1 & & \downarrow f_2 \\ \mathfrak{N}(\mathfrak{U}_1) & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) \end{array}$$

Continue in this fashion to obtain:

$$\begin{array}{ccccccccc} \mathfrak{N}(\mathfrak{U}_1) & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) & \overset{\pi_2}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_3) & \leftarrow & \cdots & P & \overset{\alpha}{\rightarrow} & M \\ \downarrow f_1 & & f_2 \downarrow & & f_3 \downarrow & & & \downarrow f & & \downarrow h \\ \mathfrak{N}(\mathfrak{U}_1) & \overset{\pi_1}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_2) & \overset{\pi_2}{\leftarrow} & \mathfrak{N}(\mathfrak{U}_3) & \leftarrow & \cdots & P & \overset{\alpha}{\rightarrow} & M \end{array}$$

Since for all i, $\mathfrak{N}(\mathfrak{U}_i)$ is a bellows, h is B-inducible.

COROLLARY 5.4. *A periodic homeomorphism h is B-inducible iff for every $\varepsilon > 0$ there exists an invariant bellow-cover of M of mesh less than ε.*

PROOF. This follows from the proof of Theorem 5.3. □

There are several ways of improving the result in Theorem 5.3 as indicated in the following

DEFINITION 5.5. A B-inducible homeomorphism h is *E-inducible* iff for all i, j, P_i and P_j of Definition 5.1 are homeomorphic. It is *T-inducible* iff each P_i is a tree. Finally, it is *ET-inducible* iff it is both E- and T-inducible.

Naturally, we have the following

QUESTION 5.6. What conditions on M, M_h and h are necessary and sufficient in order for a periodic homeomorphism h to be E-, T-, or ET-inducible?

From Corollary 5.4 we see that the existence of invariant bellows-covers of arbitrarily small mesh with homeomorphic nerves implies E-inducibility. Similarly, if these covers are tree-covers we obtain T-inducibility. Of course, in order to obtain T-inducibility M must be tree-like.

A stronger way of inducing periodic homeomorphisms can be described as follows. Recall the definition of the #-graph associated with a periodic homeomorphism (Definition 2.11).

DEFINITION 5.7. A periodic homeomorphism h with #-graph Γ is called (Γ-) *perfect* iff it is E-inducible and all the factor spaces in the inverse sequence are either Γ-bellows or $(1 + \Gamma)$-bellows.

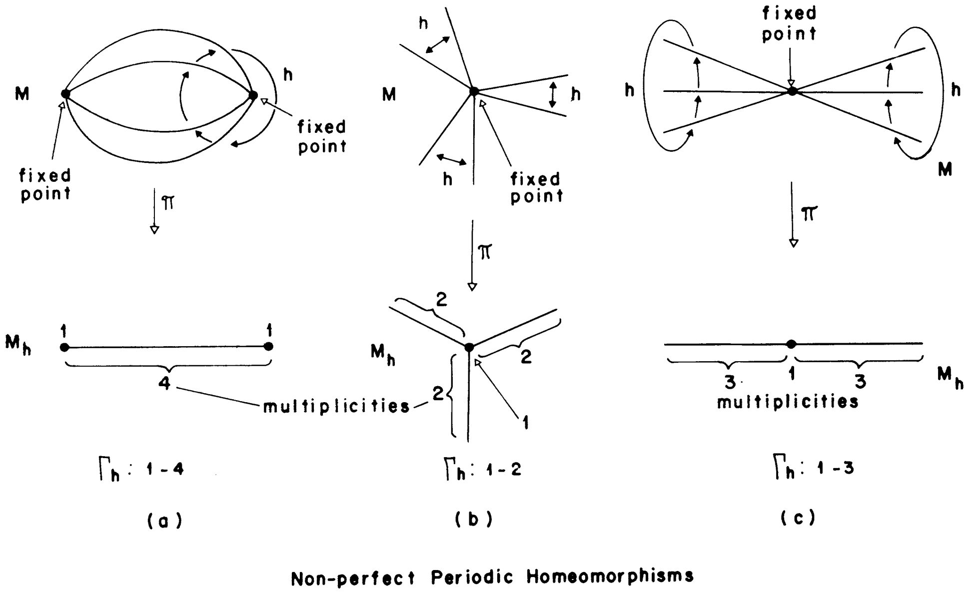


FIGURE 5.1

It is easy to find examples of perfect homeomorphisms for some admissible Γ (see Definition 2.11) as can be seen from Remark 2.21.

In Figure 5.1 we have examples of periodic homeomorphisms which are not perfect. All these homeomorphisms share the property of having orbit spaces "not related" to their #-graph. We proceed to introduce this notion more formally.

DEFINITION 5.8. Let $C = \{C_i \mid i = 1,\ldots,\mu\}$ be an open finite cover of M_h. The *natural #-graph of C* is defined as (C, σ_c) where $\sigma_c(C_i) = \min\{\text{mult}\, x \mid x \in C_i\}$. A number $\varepsilon > 0$ is called *critical for h* iff for any finite open cover C of mesh less than ε, the *set* $\{\sigma_c(C_i) \mid C_i \in C\}$ equals Γ_h.

DEFINITION 5.9. Let h be a divisible homeomorphism with #-graph Γ_h. We say that M_h is *perfect* iff for every $\varepsilon > 0$ there exist a cover C of mesh less than ε and such that $(C, \sigma_c) = 1 + \cdots + \Gamma_n$ (see Definition 2.17).

So to speak, M_h is perfect iff it is the "inverse limit of Γ_h's". Observe how the orbit spaces of the examples in Figure 5.1 are not perfect.

We want to describe a situation that forces orbit spaces to be perfect.

Standing Assumptions 5.10. In what follows, we will be assuming that M is tree-like, M_h is chainable and that h is a divisible periodic homeomorphism.

Recall from §1 that if h is divisible, then $M = \bigcup\{F_i \mid i \in \Gamma_h\}$, $F_1 = F_{n_1} \subset F_{n_2} \subset \cdots \subset M$ where Γ_h is the #-graph of h, i.e., $\Gamma_h = \{m \in Z \mid m = \text{ord}\, x \text{ for some } x \in M\} = \{1 = n_1 < n_2 < \cdots < n_p\}$ and for $i \in \Gamma_h$, $F_i = \{x \mid \text{ord}\, x \text{ divides } i\}$ is a continuum. Then $M_h = \bigcup\{\pi(F_i) \mid i \in \Gamma_h\}$ and $\pi(F_1) \subset (F_{n_2}) \subset \cdots \subset M_h$.

LEMMA 5.11. *A critical number for h always exists.*

PROOF. Let us find one. For each $1 \leqslant i \leqslant p$ let $x_i \in M_h$ be such that mult $x_i = n_i$, $n_i \in \Gamma_h$. Then $x_i \notin F_{n_{i-1}}$. Let $\varepsilon < \frac{1}{2}\min\{\text{dist}(x_i, F_{n_{i-1}}) \mid i = 2,\ldots,p\}$. We claim that ε

is a critical number for h. For if C is a chain of mesh less than ε, then, for every $1 \leqslant i \leqslant p$, there exists a $C_{s_i} \in C$ such that $x_i \in C_{s_i}$ and $C_{s_i} \cap F_{n_{i-1}} = \varnothing$. Thus $\sigma_c(C_{s_i}) = n_i$ so $\sigma_c(C) = \Gamma_h$. $\square$

LEMMA 5.12. *Let $\varepsilon > 0$ be a critical number for h and $D = \{D_i \mid i = 1, \ldots, \lambda\}$ be a chain-cover of M_h of mesh less than ε such that $D_1 \cap \pi(F_1) \neq \varnothing$. Then there exist integers $0 = q_0 < q_1 < \cdots < q_p = \lambda$ such that $\sigma_c(D_i) = n_j$ iff $q_{j-1} < i \leqslant q_j$.*

PROOF. If we denote by E_i the subcover of D that intersects $\pi(F_{n_i})$ then, since $\pi(F_{n_i})$ is connected, E_i is a subchain of D. Because $D_1 \in E_1$ and $\pi(F_1) \subset \pi(F_{n_2}) \subset \cdots \subset \pi(F_{n_p}) = M_h$, we must have that $E_1 \subset E_2 \subset \cdots \subset E_p$. Set q_i equal to the index of the last link on E_i. The lemma follows. $\square$

THEOREM 5.13. *There exists an $\varepsilon > 0$ such that if C and D are any two chain-covers of M_h of mesh less than ε with natural #-graphs σ_C and σ_D, respectively, and such that C_1 and D_1 intersect $\pi(F_1)$, then the (C, σ_c)-bellows and the (D, σ_D)-bellows are homeomorphic. Furthermore, they are homeomorphic to the Γ_h-bellows, if F_1 is a singleton, and to the $(1 + \Gamma_h)$-bellows, otherwise.*

PROOF. Let $\varepsilon > 0$ be a critical number for h such that $\varepsilon < \frac{1}{2} \operatorname{diam} \pi(F_1)$ if $\operatorname{diam} \pi(F_1) \neq 0$. By this choice of ε, Lemma 5.12 is telling us that if C is a chain cover of M_h of mesh less than ε starting at $\pi(F_1)$ then (C, σ_c) is of the form "$1 - 1 - \cdots - 1 - n_1 - n_1 - \cdots - n_1 - \cdots - n_p - n_p - \cdots - n_p$" where the number of 1's appearing is one, if F_1 is a singleton, or more than one, otherwise.

The rest follows from Lemmas 2.16 and 2.18 and by noticing that the reduced #-chain of (C, σ_c) is either Γ_h or $1 + \cdots + \Gamma_h$. $\square$

DEFINITION 5.14. A subcontinuum H of a chainable continuum M is an *end-continuum of M* iff for every $\varepsilon > 0$ there exists a chain-cover $C = \{C_i \mid i = 1, \ldots, \mu\}$ of M of mesh less than ε such that $C_1 \cap H \neq \varnothing$.

THEOREM 5.15. *Let M be a tree-like continuum and h a divisible homeomorphism such that M_h is chainable and $\pi(F_1)$ is an end-continuum of M_h. Then M_h is perfect.*

PROOF. It follows from Lemma 5.12 and the proof of Theorem 5.13. $\square$

Notice that these results not only guarantee the existence of covers of M_h of the right kind but assure it for any cover of sufficiently small mesh.

Theorem 5.15 suggests the following

QUESTION 5.16. If M is tree-like and h is divisible and such that M_h is chainable and $\pi(F_1)$ is an end-continuum, is h perfect?

Once M_h is perfect, it seems that h can also be made perfect by first lifting every cover C of M_h according to its natural #-graph σ_c and then applying Theorem 5.13. However, things do not work so nicely as we can see from the following

EXAMPLE 5.17. Consider the continuum M which is the intersection of covers following the pattern shown in Figure 5.2(a). Consider the period-2 homeomorphism with one fixed point p induced by the period-2 homeomorphisms shown at the bottom of the figure. It is not hard to see that this pair satisfies the hypothesis of Theorem 5.15. Therefore M_h is perfect.

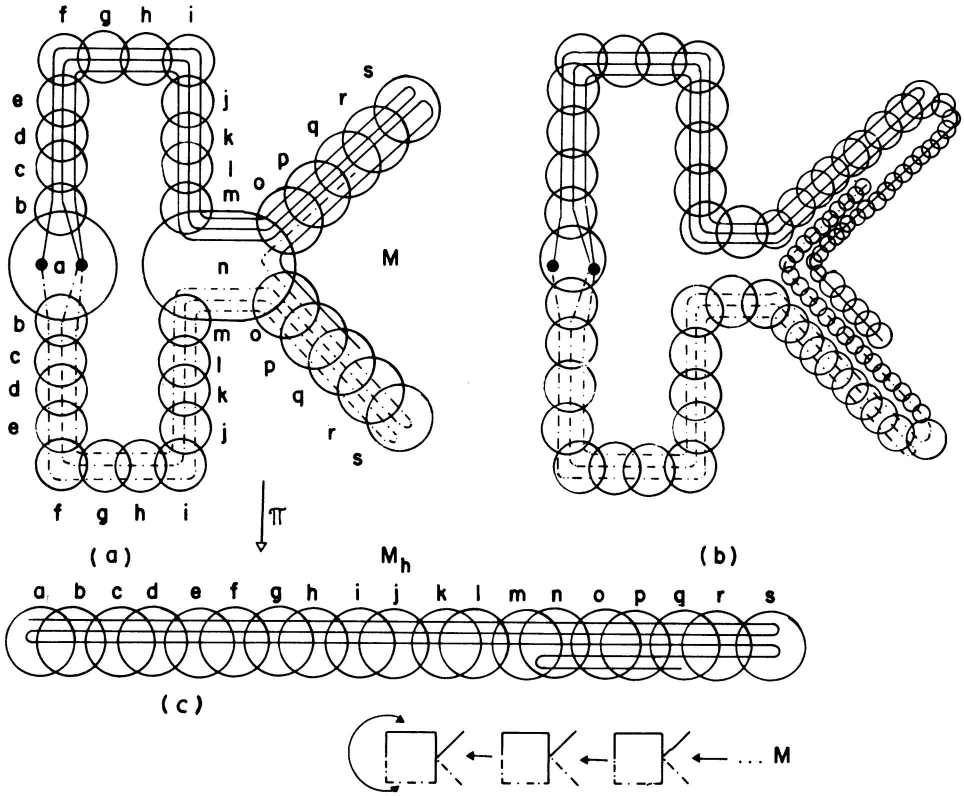


FIGURE 5.2

The chain-cover C of M_h shown in Figure 5.2(c) could be obtained by refining the projection $\pi(\mathscr{C})$ of a spread cover $\mathscr{C}$ of M, as shown in Figure 5.3. Observe then, that the coarser cover of M of Figure 5.2(a) is just $L(C, \mathscr{C})$, i.e., the lift of C induced by $\mathscr{C}$.

The natural #-graph of C takes the value 2 for all elements of C except for the link marked with "a", where it takes the value 1. However, C cannot be lifted properly since there is no possible way of 2-lifting the link marked with "n". This shows that it is important to choose the appropriate sequence of chain covers of M_h.

Despite this, h is perfect as we can see from the consolidation shown in Figure 5.2(b).

Now, let us look at the modification of this example shown in Figure 5.4(a). The arrows point to the difference. In this case it is not obvious that h is perfect since we cannot do a consolidation similar to that of Figure 5.2(b).

This new example is planar, atriodic and "x-odic" as shown in Figure 5.4(b), where a consolidation of the type of Figure 5.2(b) is presented.

It even seems possible that, by defining the embeddings more precisely, the example becomes nonchainable (by forcing it to have positive span [Le]). That would constitute a negative answer to Question 5.16.

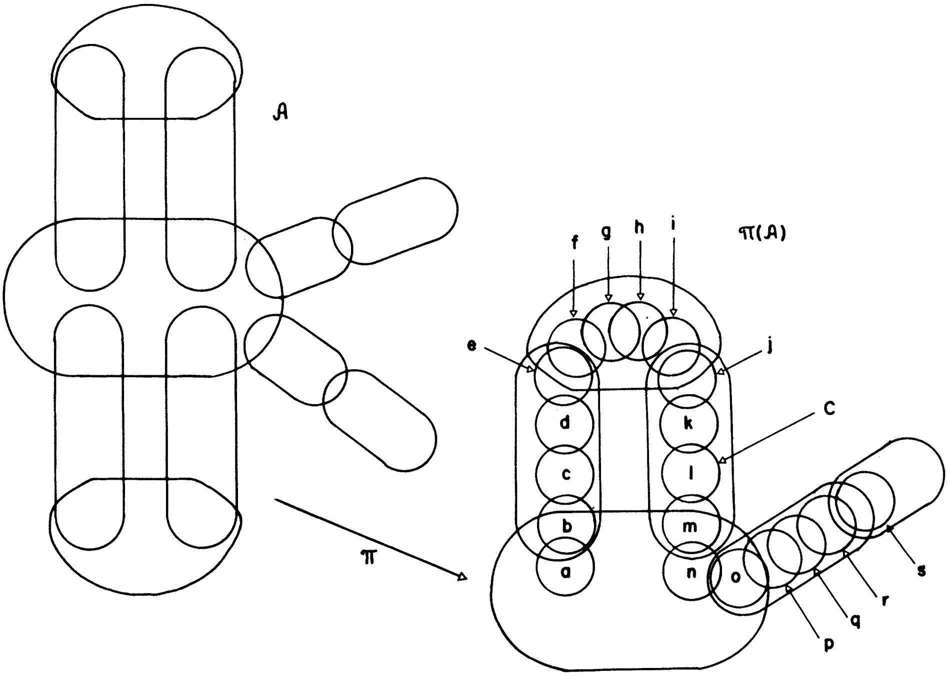


FIGURE 5.3

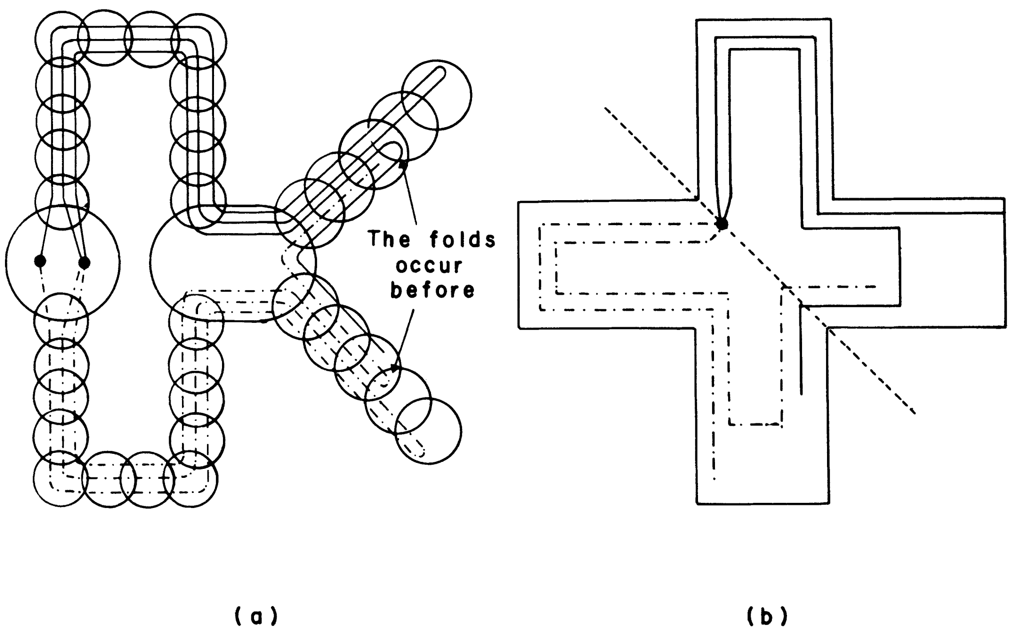


FIGURE 5.4

In [T], several perfect homeomorphisms on chainable continua were constructed. It was even suggested [T] that there possibly exist perfect homeomorphisms of all possible kinds for the pseudo-arc.

REMARK 5.18. Due to the hereditary indecomposability of the pseudo-arc, every periodic homeomorphism on it is divisible. In addition, its orbit space is also a pseudo-arc [**R**] and therefore it is homogeneous. Then, any proper subcontinuum of it (the orbit space), in particular the projection of the fixed point set, is an end-continuum.

This remark and Theorem 5.15 tell us that the orbit spaces in the case of the pseudo-arc are always perfect.

Therefore, Question 5.16 can be particularized into

QUESTION 5.19 (BRECHNER). Is every periodic homeomorphism on the pseudo-arc perfect?

A positive answer to this question would represent tremendous progress in the study of the conjugacy classes of periodic homeomorphisms of the pseudo-arc.

And, of course, we can pose the question in its full generality:

QUESTION 5.20. Given any divisible homeomorphism of a tree-like continuum M, does the condition "M_h is perfect" imply that h is perfect?

We conclude this section with two interesting observations.

REMARK 5.21. Since the existence of a perfect period-2 homeomorphism on a continuum M makes M chainable, we have that no period-2 homeomorphism on a nonchainable continuum can be perfect.

REMARK 5.22. Any conjugate of a perfect homeomorphism is perfect.

6. Conjugacy. In this section we develop some tools that represent a useful aid in the study of the conjugacy classes of periodic homeomorphisms of tree-like continua and particularly of the pseudo-arc.

In the whole section, $h_1\colon M_1 \to M_1$ and $h_2\colon M_2 \to M_2$ will denote periodic homeomorphisms on the tree-like continua M_1 and M_2. The orbit spaces will be denoted by M_{h_1} and M_{h_2} and $\pi_1\colon M_1 \to M_{h_1}$ and $\pi_2\colon M_2 \to M_{h_2}$ will denote the projection maps.

DEFINITION 6.1. Let $f\colon M_{h_1} \to M_{h_2}$ be a map. We say that *f preserves multiplicity* or is *multiplicity preserving* iff $\operatorname{mult} f(x) = \operatorname{mult} x$ for all $x \in M_{h_1}$.

DEFINITION 6.2. Let $\mathcal{D}^1$ and $\mathcal{D}^2$ be finite open covers of M_1 and M_2, respectively. We say that $\mathcal{D}^1$ and $\mathcal{D}^2$ are *h_1-h_2-isomorphic* iff for $\lambda = 1, 2$, $\mathcal{D}^\lambda$ is invariant under h_λ and there exists a simplicial homeomorphism ϕ such that the diagram

$$\begin{tikzcd}
\mathfrak{N}(\mathcal{D}^1) \arrow[r, "\phi"] \arrow[d, "\overline{h_1}"'] & \mathfrak{N}(\mathcal{D}^2) \arrow[d, "\overline{h_2}"'] \\
\mathfrak{N}(\mathcal{D}^2) \arrow[r, "\phi"'] & \mathfrak{N}(\mathcal{D}^2)
\end{tikzcd}$$

commutes, where $\bar{h}_\lambda$ is the simplicial homeomorphism induced by h_λ, $\lambda = 1, 2$.

THEOREM 6.3. *Let M_1 and M_2 be two tree-like continua and let $h_1\colon M_1 \to M_1$ and $h_2\colon M_2 \to M_2$ be two divisible periodic homeomorphisms with orbit spaces M_{h_1} and M_{h_2}. Suppose that there exists a multiplicity preserving homeomorphism $f\colon M_{h_1} \to M_{h_2}$. Then, given $\varepsilon > 0$, there exist finite open covers $\mathfrak{D}^1$ of M_1 and $\mathfrak{D}^2$ of M_2 of mesh less than ε and such that $\mathfrak{D}^1$ and $\mathfrak{D}^2$ are h_1-h_2-isomorphic.*

PROOF. During the proof, in each usage of the letter "λ", we will be assuming implicitly that the statement works for both $\lambda = 1, 2$.

Let $\pi_\lambda\colon M_\lambda \to M_{h_\lambda}$ denote the projection maps. Since M_λ is tree-like, so is M_{h_λ} (see [**M**]).

Let $\mathcal{C}^1 = \{C^{1i} \mid i = 1, \ldots, \infty\}$ be a sequence of tree-covers of M_{h_1} such that mesh $C^{1i} < \varepsilon_i$, $\lim \varepsilon_i = 0$ and $C^{1i} = \{C_j^{1i} \mid j = 1, \ldots, l_i\}$. Then $\mathcal{C}^2 = f(\mathcal{C}^1) = \{C^{2i} \mid i = 1, \ldots, \infty\}$ where $C^{2i} = \{C_j^{2i} = f(C_j^{1i}) \mid j = 1, \ldots, l_i\}$ is a sequence of tree-covers of M_{h_2} such that mesh $C^{2i} < \delta_i$ and $\lim \delta_i = 0$.

Using Theorem 4.13 find spread covers $\mathcal{A}^\lambda = \{T[a_i^\lambda] \mid i = 1, \ldots, \alpha_\lambda\}$ of M_λ of size less than ε. Then $\pi_\lambda(\mathcal{A}^\lambda)$ is a finite open cover of M_{h_λ}. Use its Lebesgue number to find $\bar{k} > 0$ such that

$$(1)\qquad C_i^{\lambda\bar{k}} \cap C_j^{\lambda\bar{k}} \neq \varnothing \text{ implies } C_i^{\lambda\bar{k}} \cup C_j^{\lambda\bar{k}} \subset \pi_\lambda\big(T[a^\lambda]\big) \text{ for some } a^\lambda \in I(\mathcal{A}^\lambda).$$

Let $W_\lambda = L(C^{\lambda\bar{k}}, \mathcal{A}^\lambda)$ (see Definition 4.15). Use the Lebesgue number of W_λ and Theorem 4.13 to find a spread cover $\mathfrak{B}^\lambda = \{T[b_i^\lambda] \mid i = 1, \ldots, \beta_\lambda\}$ of M_h such that

$$(2)\qquad \operatorname{Shr} \mathfrak{B}^\lambda < W_\lambda < \operatorname{Shr} \mathcal{A}^\lambda.$$

Therefore $\pi_\lambda(\mathfrak{B}^\lambda) < \pi_\lambda(W_\lambda) = C^{\lambda\bar{k}}$. Now use the Lebesgue number of $\pi_\lambda(\mathfrak{B}^\lambda)$ to find $k > \bar{k}$ such that

$$(3)\qquad C^{\lambda k} < \pi_\lambda(\mathfrak{B}^\lambda) < C^{\lambda\bar{k}}.$$

We will proceed now to construct the cover $\mathfrak{D}^\lambda$ of M_λ by lifting the cover $C^{\lambda k}$ in the following way. As graphs, C^{1k} and C^{2k} are isomorphic. Let us define a #-graph $\sigma\colon C^{\lambda k} \to N^+$ by

$$\sigma_i = \sigma\big(C_i^{\lambda k}\big) = \text{g.c.d.}\big(\operatorname{ord} L\big(C_i^{1k}, \mathfrak{B}^1\big), \operatorname{ord} L\big(C_i^{2k}, \mathfrak{B}^2\big)\big).$$

Since h_λ is divisible, the range of σ is a chain, and therefore $(C^{\lambda k}, \sigma)$ is a divisible #-graph.

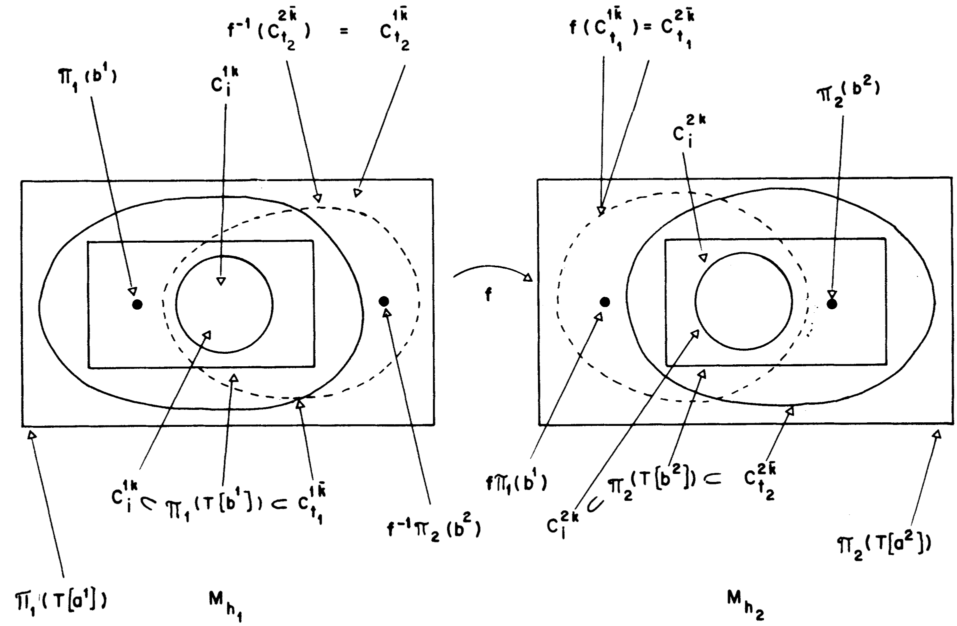


FIGURE 6.1

Now define $\mathcal{D}^\lambda = L_\sigma(C^{\lambda k}, \mathcal{B}^\lambda)$. If $L_{\sigma_i}(C_i^{\lambda k}, \mathcal{B}^\lambda) = \{D_{ij}^\lambda \,|\, j = 0,\ldots,\sigma_i - 1\}$ then

$$\mathcal{D}^\lambda = \left\{D_{ij}^\lambda \,|\, i = 1,\ldots,l_k;\, j = 0,\ldots,\sigma_i - 1\right\}.$$

By Theorem 4.24, $\mathcal{N}(\mathcal{D}^\lambda)$ is the $(C^{\lambda k}, \sigma)$-bellows. Hence by Remark 2.14, $\mathcal{N}(\mathcal{D}^1)$ and $\mathcal{N}(\mathcal{D}^2)$ are isomorphic. Indeed, the function ϕ': vrt $\mathcal{N}(\mathcal{D}^1) \to$ vrt $\mathcal{N}(\mathcal{D}^2)$ defined by $\phi'(D_{ij}^1) = D_{ij}^2$ is a graph-isomorphism. Extend ϕ' linearly to a simplicial homeomorphism ϕ: $\mathcal{N}(\mathcal{D}^1) \to \mathcal{N}(\mathcal{D}^2)$.

Let $\bar{h}_\lambda$: $\mathcal{N}(\mathcal{D}^1) \to \mathcal{N}(\mathcal{D}^1)$ be the isomorphism induced by h_λ. Since

$$\begin{aligned}\bar{h}_2\phi\left(D_{ij}^1\right) &= \bar{h}_2\left(D_{ij}^2\right) = D^2_{i(j+1 \bmod \sigma_i)}\\ &= \phi\left(D^1_{i(j+1 \bmod \sigma_i)}\right) = \phi\bar{h}_1\left(D_{ij}^1\right),\end{aligned}$$

$\mathcal{D}^1$ and $\mathcal{D}^2$ are h_1-h_2-isomorphic.

The only thing left to prove is that mesh $\mathcal{D}^\lambda < \varepsilon$.

Fix $1 \leqslant i \leqslant l_k$ (see Figure 6.1). By (3), for some t_λ with $1 < t_\lambda \leqslant l_{\bar{k}}$ and some $b^\lambda \in I(\mathcal{B}^\lambda)$,

$$C_i^{\lambda k} \subset \pi_\lambda\left(T[b^\lambda]\right) \subset C_{t_\lambda}^{\lambda \bar{k}} \tag{4}$$

where ord b^λ = ord $L(C_i^{\lambda k}, \mathcal{B}^\lambda)$. Then

$$C_i^{1k} = f^{-1}\left(C_i^{2k}\right),\quad f^{-1}\left(C_{t_2}^{2\bar{k}}\right) = C_{t_2}^{1\bar{k}} \tag{5}$$

and

$$C_i^{2k} = f\left(C_i^{1k}\right),\quad f\left(C_{t_1}^{1\bar{k}}\right) = C_{t_1}^{2\bar{k}}.$$

Thus, by (1), (4) and (5),

$$C_i^{1k} \subset C_{t_1}^{1\bar{k}} \cup C_{t_2}^{1\bar{k}} \subset \pi_1(T[a^1])$$

and

$$C_i^{2k} \subset C_{t_1}^{2\bar{k}} \cup C_{t_2}^{2\bar{k}} \subset \pi_2(T[a^2])$$

for some $a^\lambda \in \underline{I}(\mathcal{C}^\lambda)$. Observe from (4) that $\pi_1(b^1) \in C_{t_1}^{1\bar{k}}$ and $\pi_2(b^2) \in C_{t_2}^{2\bar{k}}$ so $f^{-1}\pi_2(B^2) \in C_{t_2}^{1\bar{k}}$ and $f\pi_1(b^1) \in C_{t_1}^{2\bar{k}}$. Therefore

$$\{\pi_1(b^1), f^{-1}\pi_2(b^2)\} \subset \pi_1(T[a^1]),$$
$$\{\pi_2(b^2), f\pi_1(b^1)\} \subset \pi_2(T[a^2]),$$

But

$$\operatorname{mult} f\pi_1(b^1) = \operatorname{mult} \pi_1(b^1) = \operatorname{ord} b^1 = \operatorname{ord} L(C_i^{1k}, \mathfrak{B}^1)$$

and

$$\operatorname{mult} f\pi_2(b^2) = \operatorname{mult} \pi_2(b^2) = \operatorname{ord} b^2 = \operatorname{ord} L(C_i^{2k}, \mathfrak{B}^2)$$

so by Lemma 3.9 $\operatorname{ord} a^\lambda \mid \sigma_i = \text{g.c.d.}(\operatorname{ord} b^1, \operatorname{ord} b^2)$.

We have the following situation: $C_i^{\lambda k} \subset \pi_\lambda(T[b^\lambda]) \subset \pi_\lambda(T[a^\lambda])$. The two latter sets lift to $T[b^\lambda]$ of order $\operatorname{ord} b^\lambda$ and $T[a^\lambda]$ of order $\operatorname{ord} a^\lambda$, respectively. Also $T[b^\lambda] < T[a^\lambda]$ following (2), and furthermore $\operatorname{ord} a^\lambda \mid \sigma_i \mid \operatorname{ord} b^\lambda$. Then we can apply Lemma 3.14(iii) to obtain

$$L_{\sigma i}(C_i^{\lambda k}, \mathfrak{B}^\lambda) < L(C_i^{\lambda k}, \mathcal{C}^\lambda) < \operatorname{Shr} \mathcal{C}^\lambda.$$

Since size $\mathcal{C} < \varepsilon$, mesh $L_{\sigma_i}(C_i^{\lambda k}, \mathfrak{B}^\lambda) < \varepsilon$. All we have to do now is to recall that $\mathfrak{D}^\lambda = L_\sigma(C^{\lambda k}, \mathfrak{B}^\lambda)$ to have mesh $\mathfrak{D}^\lambda < \varepsilon$. This concludes the proof. □

REMARK 6.4. Although the assumption of h_1 and h_2 being divisible was essential in the part of the proof where Theorem 4.24 was applied, it was not necessary in the proof of "mesh $\mathfrak{D}^\lambda < \varepsilon$".

Theorem 6.3 and Corollary 5.4 tell us that we can induce h_1 and h_2 using homeomorphic factor spaces $A_i \simeq B_i$:

$$\begin{array}{ccccccccc}
A_1 & \overset{\alpha_1}{\leftarrow} & A_2 & \overset{\alpha_2}{\leftarrow} & A_3 & \leftarrow & \cdots & A & \overset{\alpha}{\rightarrow} & M_1 \\
h_1^1 \downarrow & & h_1^2 \downarrow & & h_1^3 \downarrow & & & \tilde{h}_1 \downarrow & & \downarrow h_1 \\
A_1 & \underset{\alpha_1}{\leftarrow} & A_2 & \underset{\alpha_2}{\leftarrow} & A_3 & \leftarrow & \cdots & A & \overset{\alpha}{\rightarrow} & M_1
\end{array}$$

and

$$\begin{array}{ccccccccc}
B_1 & \overset{\beta_1}{\leftarrow} & B_2 & \overset{\beta_2}{\leftarrow} & B_3 & \leftarrow & \cdots & B & \overset{\beta}{\rightarrow} & M_2 \\
h_2^1 \downarrow & & h_2^2 \downarrow & & h_2^3 \downarrow & & & \tilde{h}_2 \downarrow & & \downarrow h_2 \\
B_1 & \leftarrow & B_2 & \leftarrow & B_3 & \leftarrow & \cdots & B & \overset{\beta}{\rightarrow} & M_2
\end{array}$$

Furthermore, Theorem 6.3 says that there exist homeomorphisms ϕ_i making the diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\phi_i} & B_i \\ h_1^i \downarrow & & \downarrow h_2^i \\ A_i & \xrightarrow{\phi_i} & B_i \end{array}$$

commute.

At first glance one would be tempted to take the inverse limit of the ϕ_i's and obtain a homeomorphism ϕ such that the following three-dimensional diagram

$$\begin{tikzcd}
A \arrow[rr, "\phi"] \arrow[dr, "\tilde{h}_1"] \arrow[dd, "\alpha"'] & & B \arrow[dr, "\tilde{h}_2"] \arrow[dd, "\beta"] & \\
& A \arrow[rr, "\phi"] \arrow[dd, "\alpha"] & & B \arrow[dd, "\beta"] \\
M_1 \arrow[rr, dashed, "\psi"] \arrow[dr, "h_1"] & & M_2 \arrow[dr, "h_2"] & \\
& M_1 & & M_2
\end{tikzcd}$$

commutes. Then just define ψ: $M_1 \to M_2$ by $\psi = \beta\phi\alpha^{-1}$ and observe that

$$\begin{aligned} \psi^{-1}h_2\psi &= \left(\beta\phi\alpha^{-1}\right)^{-1}h_2\left(\beta\phi\alpha^{-1}\right) = \alpha\phi^{-1}\left(\beta^{-1}h_2\beta\right)\phi\alpha^{-1} \\ &= \alpha\left(\phi^{-1}\tilde{h}_2\phi\right)\alpha^{-1} = \alpha\tilde{h}_1\alpha^{-1} = h_1. \end{aligned}$$

That is, h_1 and h_2 are conjugate.

Unfortunately, this is not always possible because in order to induce ϕ from the ϕ_i's, we need

$$\begin{array}{ccccccccc} A_1 & \xleftarrow{\alpha_1} & A_2 & \xleftarrow{\alpha_2} & A_3 & \leftarrow & \cdots & A \\ \phi_1 \downarrow & & \phi_2 \downarrow & & \phi_3 \downarrow & & & \downarrow \phi \\ B_1 & \xleftarrow{\beta_1} & B_2 & \xleftarrow{\beta_2} & B_3 & \leftarrow & \cdots & B \end{array}$$

to commute. In general the bonding maps α_i and β_i are different.

The next example shows a case where this diagram fails to commute.

Example 6.5. In Figure 6.2 we display constructions for a case satisfying the hypothesis of Theorem 6.3.

In Figure 6.2(a) we have M_1, the 3-to-1 Knaster continuum, with a 180° rotation as the period-2 homeomorphism h_1. Its orbit space is homeomorphic to M_1.

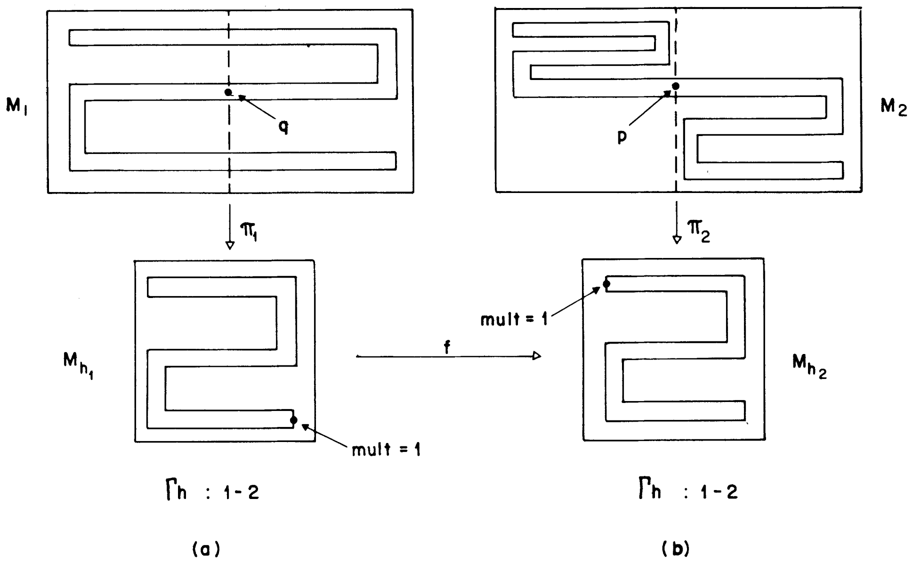


FIGURE 6.2

In Figure 6.2(b) we have M_2 defined as the intersection of the covers following the pattern shown. Again h_2 will be a 180° rotation. M_{h_2} is the 3-to-1 Knaster continuum and furthermore, there exists a multiplicity preserving homeomorphism $f\colon M_{h_1} \to M_{h_2}$.

Nevertheless, h_1 and h_2 are not conjugate since M_2 contains a cut-point (p) and M_1 does not have any, so M_1 and M_2 can not be homeomorphic.

This example suggests the following

QUESTION 6.6. Under the hypothesis of Theorem 6.3, if M_1 is homeomorphic to M_2, are h_1 and h_2 conjugate?

Now let us consider the particular case of the pseudo-arc. In Remark 5.18, we saw that every periodic homeomorphism on the pseudo-arc is divisible. Also, since the pseudo-arc is hereditarily equivalent, every subcontinuum of it is either a point or a pseudo-arc.

REMARK 6.7. If h is divisible with $\Gamma_h = \{1 = n_1 < n_2 < \cdots < n_p\}$, then, since $F_{n_p} = M$, the period of h must be n_p. Therefore the #-graph of any period-n homeomorphism on the pseudo-arc is of the form $\Gamma_h = \{1 = n_1 < n_2 < \cdots < n_p = n\}$.

THEOREM 6.8. *Let h_1 and h_2 be two period-n homeomorphisms on the pseudo-arc such that $\Gamma_{h_1} = \Gamma_{h_2}$. Let F_1^1 and F_1^2 be their respective fixed point sets. If either both F_1^1 and F_1^2 are degenerate or both are nondegenerate, then there exists a multiplicity preserving homeomorphism $f\colon M_{h_1} \to M_{h_2}$.*

PROOF. Since both h_1 and h_2 are divisible and $\Gamma_{h_1} = \Gamma_{h_2} = \Gamma$, the sets $\{\pi(F_i^\lambda) \mid i \in \Gamma\}$ are nested in M_{h_λ}, $\lambda = 1, 2$. That is, if $\Gamma = \{1 = n_1 < \cdots < n_p = n\}$ then $\pi(F_1^\lambda) \subset \pi(F_2^\lambda) \subset \cdots \subset \pi(F_n^\lambda) = M_{h_\lambda}$.

Since either both $\pi(F_1^1)$ and $\pi(F_1^2)$ are degenerate or both are nondegenerate and M_{h_λ} is a pseudo-arc, they are homeomorphic. Let f_1: $\pi(F_1^1) \to \pi(F_1^2)$ be a homeomorphism. Use Theorem 6 of [L] to extend f_1 to f_2: $\pi(F_{n_2}^1) \to \pi(F_{n_2}^2)$ and then f_2 to f_3: $\pi(F_{n_3}^1) \to \pi(F_{n_3}^2)$ and so on.

At the end we will come up with the desired multiplicity preserving homeomorphism $f = f_p$: $M_{h_1} \to M_{h_2}$. □

It follows from this theorem that any two periodic homeomorphisms of the pseudo-arc with the same #-graph and with homeomorphic fixed-point sets satisfy the hypothesis of Theorem 6.3. Then Question 6.6 can be particularized as follows.

QUESTION 6.9. If two periodic homeomorphisms of the pseudo-arc have the same #-graph and homeomorphic fixed-point sets, are they conjugate?

A positive answer to either question will narrow the number of conjugacy classes of periodic homeomorphisms associated with a specific #-graph to at most two classes.

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DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA METROPOLITANA-IZTAPALAPA, MÉXICO D. F., MEXICO